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Error Bounds for Asymptotic Solutions of Differential Equations[1]

I. The Distinct Eigenvalue Case

Frank Stenger[2]

(May 26, 1966)

The method of Olver for bounding the error term in the asymptotic solutions of a second-order equation having an irregular singularity at infinity is extended to the general system of n first-order equations in the case when the eigenvalues of the lead coefficient matrix are distinct. Vector and norm bounds are given for the difference between an actual solution vector and a partial sum of a formal solution vector. Two cases are distinguished geometrically: In one it is possible to express the error vector by a single Volterra vector integral equation; in the other it is necessary to use a simultaneous pair of Volterra vector integral equations. Some new inequalities for integral equations are given in an appendix.

1. Introduction

Let us first establish our notation. To refer to an $m \times n$ matrix $\mathbf{A}$ we shall use $\mathbf{A}=[a_{ij}]$ $(i=1, 2, \ldots, m$ $j=1,2, \ldots, n)$, while to refer to the (i, j)th element of $\mathbf{A}$ we shall use $\{\mathbf{A}\}_{ij}=a_{ij}$. We shall write $|\mathbf{A}|$ for the matrix of absolute values of the elements of $\mathbf{A}$, and for $\mathbf{A}$ and $\mathbf{B}$ real and of the same dimensions, $\mathbf{A} \leqq \mathbf{B}$ will imply that $\{\mathbf{A}\}_{ij} \leqq \{\mathbf{B}\}_{ij}$ for all (i, j). All the integration paths $\mathscr{P}$ we shall use consist of a finite number of Jordan arcs $t=t(s)$ $(a \leqq s \leqq b)$ on each of which $\frac{dt(s)}{ds}$ is continuous and non-vanishing. The notation $\sup_{t \in \mathscr{P}} |\mathbf{V}(t)|$ where $\mathbf{V}(t)$ is an $m \times n$ matrix will be used to denote the $m \times n$ matrix of non-negative numbers whose elements are the least upper bounds on $\mathscr{P}$ of the corresponding elements of $|\mathbf{V}(t)|$.

The system of differential equations we consider takes the form

$$\frac{d\mathbf{W}}{dz}=z^r\left[\sum_{k=0}^{s-1}\mathbf{A}_k z^{-k}+\mathfrak{A}_s(z)z^{-s}\right]\mathbf{W} \tag{1.1}$$

where r and s are non-negative integers. Each $\mathbf{A}_k$ is a constant $n \times n$ matrix. The n^2 elements of $\mathfrak{A}_s(z)$ are holomorphic in some domain $\mathscr{D}$ (which may be part of a Riemann surface) extending to infinity. For any path $\mathscr{P}$ extending to infinity in $\mathscr{D}$, we suppose that $\sup_{z \in \mathscr{P}} |\{\mathfrak{A}_s(z)\}_{ij}|$ is finite for $i, j=1, 2, \ldots, n$. For notational convenience we shall write $\mathfrak{A}_0(z)=\mathfrak{A}(z)$, $\mathbf{A}_{-1}=\mathbf{O}$. The integer r denotes the rank of the system (1.1).

It is known ([2][3]) that the system (1.1) has a linearly independent set of n formal vector solutions which are asymptotic expansions of actual solutions of (1.1) as $z \to \infty$ in $\mathscr{D}$. The purpose of the present paper is to establish bounds for the difference between the actual solutions and the partial sums of formal solutions. This is achieved by an extension of the error analysis developed by Olver [1] for second-order differential equations.

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[1] An invited paper based on the author's Ph.D. thesis, University of Alberta (1965).
[2] Present address: University of Michigan, Ann Arbor, Mich.
[3] Figures in brackets indicate the literature references at the end of this paper.

2. Transformations on the Original System and Formal Solutions

For purposes of obtaining error bounds we require explicit rules for constructing formal solutions. In this section we therefore construct transformations which convert (1.1) into canonical form, and then we construct formal solutions for the transformed system.

THEOREM 2.1: *There exist constant* $n \times n$ *matrices* $\mathbf{T}_0, \mathbf{T}_1, \ldots, \mathbf{T}_{r+1}$ *such that the transformation*

$$\mathbf{X} = \mathfrak{T}(z)\mathbf{W} = \sum_{k=0}^{r+1} \mathbf{T}_k z^{-k}\mathbf{W} \tag{2.1}$$

reduces (1.1) *to the canonical form*

$$\frac{d\mathbf{W}}{dz} = z^r \mathfrak{C}(z)\mathbf{W}, \tag{2.2}$$

where $\mathfrak{C}(z)$ *can be expanded in the form*

$$\mathfrak{C}(z) = z^{r+1}\sum_{k=0}^{r+1} \mathbf{C}_k z^{-k} + z^{-2}\sum_{k=0}^{s-1} \mathbf{B}_k z^{-k} + \mathfrak{B}_s(z)z^{-s}. \tag{2.3}$$

Here the $\mathbf{C}_k$ *and* $\mathbf{B}_k$ *are constant* $n \times n$ *matrices,* $\mathbf{C}_k$ *being diagonal and* $\mathfrak{B}_s(z)$ *is an* $n \times n$ *matrix whose elements are holomorphic for all sufficiently large* z *in* $\mathscr{D}$*, and uniformly bounded on a path* $\mathscr{P}$ *extending to infinity in* $\mathscr{D}$.

We prove this important theorem here in order to furnish an explicit algorithm for computing the coefficient matrices $\mathbf{T}_k$ $(k = 0, 1, \ldots, r+1)$. For an earlier proof see Birkoff [3].

Since $\mathbf{A}_0$ has distinct diagonal elements there exists a nonsingular matrix $\mathbf{T}_0$ such that $\mathbf{T}_0^{-1}\mathbf{A}_0\mathbf{T}_0 = \mathbf{C}_0$, where $\mathbf{C}_0$ is a diagonal matrix with distinct diagonal elements. Accordingly we assume that the transformation $\mathbf{X} = \mathbf{T}_0\mathbf{W}$ has been made and that $\mathbf{A}_0$ is already in diagonal form. Then we may take $\mathbf{T}_0 = \mathbf{I}$ in (2.1). On making the transformation (2.1) in (2.2), we obtain

$$\frac{d\mathbf{W}}{dz} = z^r \mathfrak{T}^{-1}(z)\left[\mathfrak{A}(z)\mathfrak{T}(z) - z^{-r}\frac{d\mathfrak{T}(z)}{dz}\right]\mathbf{W}. \tag{2.4}$$

We choose the matrices $\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_{r+1}$ so that the matrix

$$\mathfrak{C}(z) = \mathfrak{T}^{-1}(z)\left[\mathfrak{A}(z)\mathfrak{T}(z) - z^{-r}\frac{d\mathfrak{T}(z)}{dz}\right] \tag{2.5}$$

has the form (2.3). For this purpose we need only consider the coefficients of $z^{-1}, z^{-2}, \ldots, z^{-r-1}$ in the equation

$$\mathfrak{T}(z)\mathfrak{C}(z) = \mathfrak{A}(z)\mathfrak{T}(z) - z^{-r}\frac{d\mathfrak{T}(z)}{dz}. \tag{2.6}$$

Expanding (2.6) and equating equal powers of z, we obtain

$$\sum_{k=0}^{l} (\mathbf{T}_{l-k}\mathbf{C}_k - \mathbf{A}_k\mathbf{T}_{l-k}) = \mathbf{O} \qquad (l = 0, 1, \ldots, r+1). \tag{2.7}$$

With $l = 0$, (2.7) gives $\mathbf{C}_0 = \mathbf{A}_0$. With $l = 1$ we have

$$\mathbf{T}_1\mathbf{C}_0 - \mathbf{A}_0\mathbf{T}_1 + \mathbf{C}_1 - \mathbf{A}_1 = \mathbf{O}. \tag{2.8}$$

Now for arbitrary $\mathbf{T}_1$, the diagonal part of $\mathbf{T}_1\mathbf{C}_0-\mathbf{A}_0\mathbf{T}_1=\mathbf{T}_1\mathbf{A}_0-\mathbf{A}_0\mathbf{T}_1$ is zero. Thus $\mathbf{C}_1=\text{diag}\,(\mathbf{A}_1)$. The off-diagonal elements of $\mathbf{T}_1$ can be determined by the solution of

$$\{\mathbf{T}_1\}_{ij}(\{\mathbf{A}_0\}_{jj}-\{\mathbf{A}_0\}_{ii})=\{\mathbf{A}_1\}_{ij}$$

under the assumption that the diagonal elements of $\mathbf{A}_0$ are distinct. We set the undeterminable diagonal elements of $\mathbf{T}_1$ equal to zero.

Assume that for some positive integer $l\leqq r$ we have solved for $\mathbf{C}_1, \ldots, \mathbf{C}_l$ and the off-diagonal elements of $\mathbf{T}_1, \ldots, \mathbf{T}_l$ and that we have set the diagonal elements of $\mathbf{T}_1, \ldots, \mathbf{T}_l$ equal to zero. Then from (2.7) we have

$$\mathbf{T}_{l+1}\mathbf{A}_0-\mathbf{A}_0\mathbf{T}_{l+1}+\sum_{k=1}^{l}(\mathbf{T}_{l+1-k}\mathbf{C}_k-\mathbf{A}_k\mathbf{T}_{l+1-k})+\mathbf{C}_{l+1}-\mathbf{A}_{l+1}=\mathbf{O}. \tag{2.9}$$

Again, the diagonal part of $\mathbf{T}_{l+1}\mathbf{A}_0-\mathbf{A}_0\mathbf{T}_{l+1}$ is zero. Thus the diagonal part of

$$\sum_{k=1}^{l}(\mathbf{T}_{l+1-k}\mathbf{C}_k-\mathbf{A}_k\mathbf{T}_{l+1-k})+\mathbf{C}_{l+1}-\mathbf{A}_{l+1} \tag{2.10}$$

is also zero. This enables us to determine $\mathbf{C}_{l+1}$. The off-diagonal elements of $\mathbf{T}_{l+1}$ are determinable from the scalar equation

$$\{\mathbf{T}_{l+1}\}_{ij}(\{\mathbf{A}_0\}_{jj}-\{\mathbf{A}_0\}_{ii})+\left\{\sum_{k=1}^{l}(\mathbf{T}_{l+1-k}\mathbf{C}_k-\mathbf{A}_k\mathbf{T}_{l+1-k})\right\}_{ij}=0 \tag{2.11}$$

since the diagonal elements of $\mathbf{A}_0$ are distinct. The diagonal elements of $\mathbf{T}_{l+1}$ are again not determinable and we set these equal to zero. This proves by induction that each $\mathbf{T}_k$ and $\mathbf{C}_k$ ($k=0, 1, \ldots, r+1$) can be determined as stated in the theorem.

To determine $\mathbf{B}_k$ and $\mathfrak{B}_s(z)$, we again expand (2.6) and equate coefficients of z^{-r-1-k}, $k\geqq 1$. We obtain

$$\mathbf{B}_k=\mathbf{A}_{k+r+2}-\sum_{l=1}^{\min(k,\,r+1)}(\mathbf{T}_l\mathbf{B}_{k-l}-\mathbf{A}_{k+r+2-l}\mathbf{T}_l)$$

$$-\sum_{l=k+1}^{r+1}(\mathbf{T}_l\mathbf{C}_{k+r+2-l}-\mathbf{A}_{k+r+2-l}\mathbf{T}_l)+k\mathbf{T}_k \tag{2.12}$$

for all $k\geqq 0$, where we define $\mathbf{T}_k=0$ for $k>r+1$. Since the elements of $\mathfrak{A}(z)$, $\mathfrak{T}(z)$, $\mathfrak{T}^{-1}(z)$ are holomorphic and bounded for all sufficiently large z in $\mathscr{D}$, the elements of $\mathfrak{B}_s(z)$ are similarly holomorphic and bounded.

This completes the proof of Theorem 2.1.

THEOREM 2.2: *The system* (2.2), *in which* $\mathfrak{C}(z)$ *has the canonical form* (2.3), *possesses a formal independent series solution matrix of the form*[4]

$$\widetilde{\mathbf{W}}(z)=\widetilde{\mathbf{U}}(z)\exp\,\mathfrak{Q}(z) \tag{2.13}$$

[4]Here and henceforth $\exp\,\mathbf{B}=\sum_{k=0}^{\infty}\frac{\mathbf{B}^k}{k!}$ for any square matrix $\mathbf{B}$.

where $\mathfrak{Q}(z)$ is diagonal and has the form

$$\mathfrak{Q}(z)=\sum_{k=0}^{r}\frac{\mathbf{Q}_k}{r+1-k}z^{r+1-k}+\mathbf{Q}_{r+1}\ln z=\sum_{k=0}^{r}\frac{\mathbf{C}_k z^{r+1-k}}{r+1-k}+\mathbf{C}_{r+1}\ln z \tag{2.14}$$

and[5]

$$\mathbf{U}(z)\hat{=}\mathbf{I}+\sum_{k=1}^{\infty}\mathbf{U}_k z^{-k}. \tag{2.15}$$

PROOF: Substituting (2.13) into (2.2), canceling $\exp \mathfrak{Q}(z)$ and equating coefficients of equal powers of z^{-1}, we find that

$$\sum_{j=0}^{\min(k,r+1)}(\mathbf{U}_{k-j}\mathbf{Q}_j-\mathbf{C}_j\mathbf{U}_{k-j})-(k-r-1)\mathbf{U}_{k-r-1}-\sum_{j=0}^{k-r-2}\mathbf{B}_j\mathbf{U}_{k-r-2-j}=\mathbf{O}(k=0,\ 1,\ 2,\ \ldots) \tag{2.16}$$

provided that we define $\mathbf{U}_j=\mathbf{O}$ for $j<0$. The proof of Theorem 2.2 will be complete after we prove the following lemma.

LEMMA 2.1: *With $\mathbf{U}_0=\mathbf{I}$ and $\mathfrak{Q}(z)$ diagonal, equation* (2.16) *uniquely determines $\mathbf{U}_j$ and $\mathbf{Q}_j$, $j=0,\ 1,\ 2,\ \ldots$.*

PROOF: With $k=0$ we have

$$\mathbf{Q}_0=\mathbf{C}_0. \tag{2.17}$$

With $k=1$, we have

$$\mathbf{U}_1\mathbf{C}_0-\mathbf{C}_0\mathbf{U}_1+\mathbf{Q}_1-\mathbf{C}_1=\mathbf{O}. \tag{2.18}$$

Now if $\mathbf{U}_1$ is an arbitrary $n\times n$ matrix, the diagonal part of $\mathbf{U}_1\mathbf{C}_0-\mathbf{C}_0\mathbf{U}_1$ is zero. By assumption, $\mathbf{Q}_1$ and $\mathbf{C}_1$ are diagonal; hence $\mathbf{Q}_1=\mathbf{C}_1$. Since the diagonal elements of $\mathbf{C}_0$ are distinct, the off-diagonal elements of $\mathbf{U}_1$ given by the solution of $\{\mathbf{U}_1\}_{ij}(\{\mathbf{C}_0\}_{jj}-\{\mathbf{C}_0\}_{ii})=0$ are zero. The diagonal elements of $\mathbf{U}_1$ are not determinable at this point and all we can say here is that $\mathbf{U}_1=\mathbf{D}_1$, where $\mathbf{D}_1$ is an arbitrary diagonal matrix.

Assume that for some integer k in the range $1\leqq k\leqq r$ we have found $\mathbf{Q}_j=\mathbf{C}_j$ and $\mathbf{U}_j=\mathbf{D}_j$ where $\mathbf{D}_j$ is an arbitrary diagonal matrix and $1\leqq j\leqq k$. Then solving for $\mathbf{U}_{k+1}$ and $\mathbf{Q}_{k+1}$ we have from (2.16)

$$\mathbf{U}_{k+1}\mathbf{C}_0-\mathbf{C}_0\mathbf{U}_{k+1}+\sum_{j=1}^{k}(\mathbf{D}_{k+1-j}\mathbf{C}_j-\mathbf{C}_j\mathbf{D}_{k+1-j})+\mathbf{Q}_{k+1}-\mathbf{C}_{k+1}=\mathbf{O} \tag{2.19}$$

or, since diagonal matrices commute,

$$\mathbf{U}_{k+1}\mathbf{C}_0-\mathbf{C}_0\mathbf{U}_{k+1}+\mathbf{Q}_{k+1}-\mathbf{A}_{k+1}=\mathbf{O}. \tag{2.20}$$

On comparing (2.20) with (2.18) it follows by exactly similar arguments to those used for (2.18) that $\mathbf{Q}_{k+1}=\mathbf{C}_{k+1}$ and that $\mathbf{U}_{k+1}=\mathbf{D}_{k+1}$, where $\mathbf{D}_{k+1}$ is an arbitrary diagonal matrix. We have thus proved by induction that for every integer k in the range $0\leqq k\leqq r+1$, $\mathbf{Q}_k=\mathbf{C}_k$ and $\mathbf{U}_k=\mathbf{D}_k$, where the $\mathbf{D}_k$ are undetermined diagonal matrices.

[5] The symbol "$\hat{=}$" here and henceforth denotes a formal equality.

With $k=r+2$, we have, from (2.16) and the above results,

$$\mathbf{U}_{r+2}\mathbf{C}_0-\mathbf{C}_0\mathbf{U}_{r+2}+\sum_{j=1}^{r+1}(\mathbf{D}_{r+2-j}\mathbf{C}_j-\mathbf{C}_j\mathbf{D}_{r+2-j})+\mathbf{B}_0\mathbf{D}_1=\mathbf{O}. \tag{2.21}$$

or, since the sum on the left is a sum of diagonal matrices,

$$\mathbf{U}_{r+2}\mathbf{C}_0-\mathbf{C}_0\mathbf{U}_{r+2}-\mathbf{B}_0-\mathbf{D}_1=\mathbf{O}. \tag{2.22}$$

The diagonal part of $\mathbf{U}_{r+2}\mathbf{C}_0-\mathbf{C}_0\mathbf{U}_{r+2}$ is zero; hence $\mathbf{D}_1=\mathbf{U}_1=-\operatorname{diag}\mathbf{B}_0$. The off-diagonal elements of $\mathbf{U}_{r+2}$ are given by the solution of $\{\mathbf{U}_{r+2}\}_{ij}(\{\mathbf{C}_0\}_{jj}-\{\mathbf{C}_0\}_{ii})=\{\mathbf{B}_0\}_{ij}$. The diagonal part $\mathbf{D}_{r+2}$ of $\mathbf{U}_{r+2}$ cannot be determined at this point.

Assume that for some positive integer l, the matrices $\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_l$ as well as the off-diagonal elements of the matrices $\mathbf{U}_{l+1}, \mathbf{U}_{l+2}, \ldots, \mathbf{U}_{r+l+1}$ have been determined while the diagonal matrices consisting of the diagonals of $\mathbf{U}_k (l+1 \leqq k \leqq r+l+1)$ are still undetermined. Then, from (2.16) we have

$$\mathbf{U}_{r+l+2}\mathbf{C}_0-\mathbf{C}_0\mathbf{U}_{r+l+2}+\sum_{j=1}^{r+1}(\mathbf{U}_{r+2+l-j}\mathbf{C}_j-\mathbf{C}_j\mathbf{U}_{r+l+2-j})+\sum_{j=1}^{l+1}\mathbf{B}_{j-1}\mathbf{U}_{l+1-j}-(l+1)\mathbf{U}_{l+1}=\mathbf{O}. \tag{2.23}$$

On writing $\mathbf{U}_j=\overline{\mathbf{U}}_j+\mathbf{D}_j$, $j>l$, where the diagonal elements of $\overline{\mathbf{U}}_j$ are zero, and using the fact that diagonal matrices commute, we find that

$$\overline{\mathbf{U}}_{r+l+2}\mathbf{C}_0-\mathbf{C}_0\overline{\mathbf{U}}_{r+l+2}+\sum_{j=1}^{r+1}(\overline{\mathbf{U}}_{r+l+2-j}\mathbf{C}_j-\mathbf{C}_j\overline{\mathbf{U}}_{r+l+2-j})-\sum_{j=1}^{l+1}\mathbf{B}_{j-1}\mathbf{U}_{l+1-j}-(l+1)(\overline{\mathbf{U}}_{l+1}+\mathbf{D}_{l+1})=\mathbf{O}. \tag{2.24}$$

Thus each $\mathbf{U}_j$, $j \geqq 1$, is uniquely determined. This completes the proof of Lemma 2.1 and of Theorem 2.2.

Clearly, the above procedure can be modified to compute a particular formal series solution vector independently of the others; the jth series solution vector is

$$\widetilde{\mathbf{W}}_j(z)=\widetilde{\mathbf{U}}_j(z)e^{q_j(z)} \tag{2.25}$$

where $\widetilde{\mathbf{U}}_j(z)$ is the jth vector in $\widetilde{\mathbf{U}}(z)$ and $q_j(z)=\{\mathfrak{Q}(z)\}_{jj}$. For later convenience we set

$$q_j(z)=\sum_{k=0}^{r}\frac{q_{jk}}{r+1-k}z^{r+1-k}+q_{j,\,r+1}\log z. \tag{2.26}$$

The representation (2.25) *illustrates that we have a one-to-one correspondence between the eigenvalues of* $\mathbf{A}_0$ *and the formal vector solutions.*

3. Error Bounds for the Formal Partial Sum Approximation

3.1. The Differential Equation for an Approximation

We denote the jth column of $\mathbf{U}_k$ ($k=0,1,\ldots$) as defined by Theorem 2.2 by $\mathbf{U}_{jk}$. Starting with the partial sum

$$\Phi_{jm}(z)=\left(\sum_{k=0}^{m-1}\mathbf{U}_{jk}z^{-k}\right)e^{q_j(z)} \tag{3.1}$$

we define a vector $\mathbf{R}_{jm}(z)$ by the differential equation

$$\frac{d\Phi_{jm}(z)}{dz}-z^r\mathfrak{C}(z)\Phi_{jm}(z)=\mathbf{R}_{jm}(z)e^{q_j(z)} \tag{3.2}$$

where $\mathfrak{C}(z)$ is defined by (2.3).

Expanding (3.2) and using equations (2.26) and (2.16), we obtain

$$\mathbf{R}_{jm}(z)=-\sum_{\mu=0}^{r+m+1}\left[\sum_{k=0}^{\min(\mu,\,r+1)}(\mathbf{Q}_k-\mathbf{I}q_{jk})\mathbf{U}^{\dagger}_{\mu-k}+(\mu-r-1)\mathbf{U}^{\dagger}_{j,\,\mu-r-1}\right.$$

$$\left.+\sum_{k=0}^{\mu-r-2}\mathbf{B}_k\mathbf{U}^{\dagger}_{j,\,\mu-r-2-k}\right]z^{r-\mu}-\sum_{k=0}^{m-1}(\mathfrak{B}_{m-k}(z)\mathbf{U}^{\dagger}_{jk})z^{-m-2} \tag{3.3}$$

where $\mathbf{U}^{\dagger}_{jk}=\mathbf{U}_{jk}$ if $0\leqq k\leqq m-1$; $\mathbf{O}$ otherwise.

Consider the ith element $\mathbf{R}_{i\,jm}(z)$ of $\mathbf{R}_{jm}(z)$. If $i=j$ then using (2.16) again we obtain

$$\mathbf{R}_{j\,jm}(z)=m\mathbf{U}_{j\,jm}z^{-m-1}-\sum_{k=0}^{m-1}\{(\mathfrak{B}_{m-k}(z)\,\mathbf{U}_{jk})\}_j\,z^{-m-2} \tag{3.4}$$

where $\{\mathbf{V}_j\}_i$ denotes the ith element of the vector $\mathbf{V}_j$.

We also define

$$q_{ij}(z)=q_i(z)-q_j(z). \tag{3.5}$$

For $i\neq j$, we again make use of (2.16) in (3.3) to obtain

$$\mathbf{R}_{i\,jm}(z)=q'_{ij}(z)\mathbf{U}_{i\,jm}z^{-m}-z[q'_{ij}(z)-(q_{i0}-q_{j0})z^r]\mathbf{U}_{i\,jm}z^{-m-1}$$

$$-\sum_{\mu=m+1}^{m+r+1}\left[\sum_{k=\mu+1-m}^{\min(\mu,\,r+1)}(q_{ik}-q_{jk})\mathbf{U}_{ij,\,\mu-k}+(\mu-r-1)\mathbf{U}_{ij,\,\mu-r-1}\right.$$

$$\left.+\sum_{k=0}^{\mu-r-2}\{\mathbf{B}_k\mathbf{U}_{j,\,\mu-r-2-k}\}_i\right]z^{r-\mu}-\sum_{k=0}^{m-1}\{\mathfrak{B}_{m-k}(z)\mathbf{U}_{jk}\}_i\,z^{-m-2} \tag{3.6}$$

where $\mathbf{U}_{i\,jk}$ is the ith element of the vector $\mathbf{U}_{jk}$.

3.2. Two Possible Cases

In this section we shall set up integral equations for the error vector, that is, the difference between an actual solution vector of (2.2) and a partial sum of the formal solution vector (2.13).

Our aim is to construct actual solution vectors for which the formal solution vectors obtained in section 2 are asymptotic expansions, as $|z| \to \infty$. For this reason, and for purposes of obtaining good error bounds we assume $\mathcal{D}$ is such that we can choose our fixed end-points of integration, which we denote by ζ_k, at ∞. We shall show that under these conditions it is possible to express the error vector by at most a simultaneous pair of Volterra vector integral equations.

The error bounds we shall obtain are considerably sharper in the case when we can express the error by a single Volterra vector integral equation than in the case when we require a simultaneous pair of Volterra vector integral equations. It is thus desirable to use a single equation whenever possible.

We have already noted at the end of section 2 that with our construction of formal solutions there is a one-to-one correspondence between the eigenvalues of $\mathbf{C}_0$ and the vector solutions. Let λ_j be an eigenvalue of $\mathbf{C}_0$.

DEFINITION 3.1: *The eigenvalue λ_j will be called an extreme eigenvalue if there exists a path $\mathcal{P}$ lying in $\mathcal{D}$, joining some point $z(|z|<\infty)$ in $\mathcal{D}$ with $\zeta_k(|\zeta_k|=\infty)$ such that* Re $q_{ij}(t)$ *(see equation* (3.1)*) increases monotonically*[6] *as* t *traverses $\mathcal{P}$ from* z *to ζ_k, for* $i=1, 2, \ldots, n$. *Otherwise λ_j will be called an interior eigenvalue.*

Consider the eigenvalues of $\mathbf{C}_0$ as points in the complex plane enclosed by the smallest possible closed convex polygon Π. Except for rotation, the polygon Π and the differences between the various eigenvalues of $\mathbf{A}_0$ are left invariant under the transformations $\mathbf{W}=\mathbf{X}e^{\lambda z^{r+1}}$, $\mathbf{W}=\mathfrak{T}(z)\mathbf{X}$ ($\mathfrak{T}(z)$ as in Theorem 2.1), $z=\omega\zeta$, ω a constant such that $|\omega|=1$. Thus we may assume, without loss of generality, that the vector solution of interest corresponds to the eigenvalue $\lambda_j=0$. Moreover, in the case when zero is an extreme point of the polygon Π, we may assume that all other eigenvalues of $\mathbf{A}_0$ are within the closed sector $|\arg\lambda| \leqq \alpha \leqq \frac{\pi}{2}$ and that there is at least one eigenvalue on each of the rays $\arg\lambda=\alpha$ and $\arg\lambda=-\alpha$. In the case when zero is an interior point of Π, we assume that no eigenvalue (other than the zero eigenvalue) is located on the imaginary axis, that there are some eigenvalues of $\mathbf{A}_0$ in each of the sectors $|\arg\lambda| \leqq \alpha$ and $|\pi-\arg\lambda| \leqq \alpha$, $0 \leqq \alpha < \frac{\pi}{2}$, and that there is at least one eigenvalue on each of the lines $\arg\lambda=\alpha$ and $\arg\lambda=-\alpha$.

Suppose for example that we have the case indicated in figure 1, that $\mathfrak{V}(z)$ is regular for all $|z|>\rho$, that $r=0$ and that $q_{i,r+1}=q_{i1}=0$ $(i=1, 2, 3, j)$. It is then clear that $|e^{\lambda_i z}|$ $(i=1, 2, 3, j)$ increases monotonically along any path for which $\left|\arg\frac{dz}{dx}\right| \leqq \frac{\pi}{2}-\alpha$. Thus assuming the points λ_i $(i=1, 2, 3, j)$ to be in the z-plane as indicated in figure 1, we can, for example, connect any point

[6] For any two points z_1, z_2 in the order z, z_1, z_2, ζ_k on $\mathcal{P}$ we have Re $q_{ij}(z_1) \leqq$ Re $q_{ij}(z_2)$.

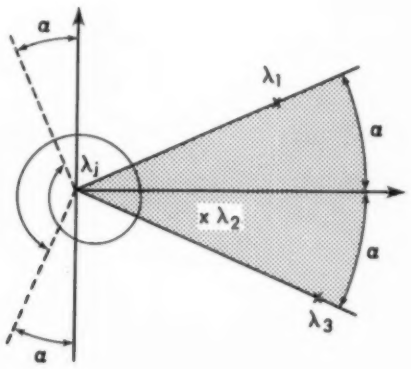

FIGURE 1.

in the interior of the overlapping sector $|\arg z| \leqq \frac{3}{4}\pi - \alpha - \delta$, $0 \leqq \delta < \frac{\pi}{4}$ of figure 1 with $+\infty$ by a path $\mathscr{P}$ consisting of at most three straight lines along each of which $\left|\arg \frac{dz}{dx}\right| \leqq \frac{\pi}{2} - \alpha$ and none of which passes through the origin.

For arbitrary rank $r+1$ it follows that, since $|\arg q_{ij}(t) - \arg [(q_{io} - q_{jo})t^{r+1}/(r+1)]|$ can be made arbitrarily small by taking $|t|$ sufficiently large, we can connect any z ($|z|$ sufficiently large) in

$$\left|\arg z - \frac{2\pi k}{r+1}\right| \leqq \frac{1}{r+1}\left(\frac{3\pi}{2} - \alpha - \delta\right)\left(0 \leqq \delta < \frac{\pi}{4}\right), \ (k = 0, 1, \ldots, r)$$

to

$$\zeta_k = \infty \exp\left[\frac{2\pi k\sqrt{-1}}{r+1}\right]$$

by a path $\mathscr{P}$ along which Re $q_{ij}(t)$ increases monotonically for $i = 1, 2, \ldots, n$, provided that $\alpha < \frac{\pi}{2}$. In the case when $\alpha = \frac{\pi}{2}$ there may still be a region between two parallel straight lines (no longer a sector of positive angle) in which there is at least one path $\mathscr{P}$ joining z with ζ_k along which Re $q_{ij}(t)$ increases monotonically for all i.

Let us now consider a case in which λ_j is an interior point of the polygon. If the eigenvalues of $\mathbf{C}_0$ are as indicated in figure 2 then, given any number δ in $0 \leqq \delta < \pi - \alpha$ there is a path joining $-\infty$, any point z in the sector $\left|\arg z - \frac{\pi}{2}\right| \leqq \pi - \alpha - \delta$ and $+\infty$, consisting of at most four straight lines (none of which passes through the origin) such that Re $(\lambda_1 t)$, Re $(\lambda_2 t)$ increase monotonically while Re $(\lambda_3 t)$ decreases monotonically as t traverses $\mathscr{P}$ from $-\infty$ to ∞. If we require that a neighborhood of the origin is to be avoided by $\mathscr{P}$, $|z|$ may need to be taken large when δ is small. Similarly, it is easy to see that there is such a path joining $-\infty$, any point z in the sector $|\arg z + \frac{1}{2}\pi| \leqq \pi - \alpha - \delta$ and $+\infty$. Thus these two families of paths, one of which passes around the origin in a positive sense and one in the negative sense as t traverses $\mathscr{P}$ from $-\infty$ to $+\infty$, cover the complete neighborhood of infinity.

In the general case when λ_j is an interior eigenvalue of Π, we divide the integers $\mathbf{N} = 1, 2, \ldots, n$ into two disjoint classes $\mathbf{N}_1$ and $\mathbf{N}_2$ such that if λ_i is in $|\arg \lambda| \leqq \alpha$ then $i \in \mathbf{N}_1$ while if λ_i is in $|\pi - \arg \lambda| \leqq \alpha$ then $i \in \mathbf{N}_2$. Note that j may be taken either in $\mathbf{N}_1$ or in $\mathbf{N}_2$. Clearly there are two families of paths connecting $\zeta_k^{(1)} = \exp[(2k \pm 1)\pi\sqrt{-1}/(r+1)]$, and $\zeta_k^{(2)} = \exp[2k\pi\sqrt{-1}/(r+1)]$

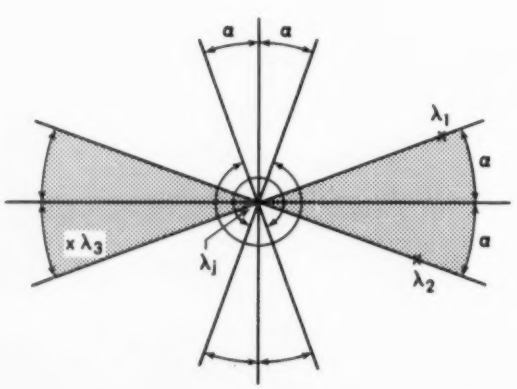

FIGURE 2.

$(k=0, 1, \ldots, n)$ such that as t traverses $\mathscr{P}$ from $\zeta_k^{(1)}$ to $\zeta_k^{(2)}$ $\operatorname{Re} q_{ij}(t)$ decreases or increases monotonically according as $i \in \mathbf{N}_1$ or $i \in \mathbf{N}_2$. These two families of paths cover the sectors $\left|\arg z-(2k \pm \frac{1}{2})\frac{\pi}{r+1}\right| \leqq \frac{\pi-\alpha}{r+1}$.

THEOREM 3.1: *Let λ_j be an eigenvalue of $\mathbf{C}_0$. If λ_j is a vertex of Π, then it is an extreme eigenvalue; if it is an interior point of Π then it is an interior eigenvalue.*

Let $\mathfrak{S}(z)$ be holomorphic in a domain $\mathscr{D}$ extending to infinity in the direction $\arg z = 2\pi k/(r+1)$, for some integer k in $0 \leqq k \leqq r$. Let $\mathscr{E}$ be the smallest closed disc with center at the origin containing the zeros of $q_{ij}'(t)\,(i=1, 2, \ldots, n;\ i \neq j4$, and let $\mathscr{D}^* = \mathscr{D} - \mathscr{E} - \{\infty\}$.[7] We define a region $\mathscr{D}(z, \zeta_k)$ to be the union of all points z such that there is a path $\mathscr{P}$ connecting z and $\zeta_k = \exp[2\pi k\sqrt{-1}/(r+1)]$ satisfying the following *extreme eigenvalue conditions*:[8]

(1) Except for ζ_k, $\mathscr{P}$ lies entirely in $\mathscr{D}^*$;

(2) For any two points t_1 and t_2 in the order ζ_k, t_1, t_2, z on $\mathscr{P}$, we have

$$|\exp[q_{ij}(t_2)-q_{ij}(t_1)]| \leqq 1 \qquad (i=1, 2, \ldots, n);$$
$$\mathscr{V}_{\mathscr{P}}(t^{-1}) = \int_{\mathscr{P}} |t^{-2}dt| \text{ is bounded.}$$

The variation symbol introduced in condition (3) above is more generally defined as follows: If $\mathbf{F} = \mathbf{F}(z)$ is a vector of holomorphic functions of z, we define

$$\mathscr{V}_{\mathscr{P}}(\mathbf{F}) = \int_{\mathscr{P}} |d\mathbf{F}| = \int_{\mathscr{P}} \left|\frac{d\mathbf{F}}{dz}\,dz\right|. \tag{3.7}$$

Let us set

$$\mathbf{D}_j(z) = \mathfrak{Q}(z) - q_j(z)\mathbf{I}. \tag{3.8}$$

If a vector $\mathbf{W}_j(z)$ satisfies (2.2), then by (3.2) the error vector

$$\boldsymbol{\epsilon}_{jm}(z) = \{\mathbf{W}_j(z) - \Phi_{jm}(z)\}e^{-q_j(z)} \tag{3.9}$$

satisfies

$$\frac{d}{dz}\boldsymbol{\epsilon}_{jm}(z) - \mathbf{D}_j(z)\boldsymbol{\epsilon}_{jm}(z) = \frac{1}{z^2}\mathfrak{B}(z)\boldsymbol{\epsilon}_{jm}(z) - \mathbf{R}_{jm}(z). \tag{3.10}$$

where $\mathfrak{B}(z) = \mathfrak{B}_0(z)$. Now if $\boldsymbol{\epsilon}_{jm,k}(z)$ is a solution of

$$\boldsymbol{\epsilon}_{jm,k}(z) = \int_{\zeta_k}^{z} e^{\mathbf{D}_j(z)-\mathbf{D}_j(t)}[t^{-2}\mathfrak{B}(t)\boldsymbol{\epsilon}_{jm,k}(t) - \mathbf{R}_{jm,k}(t)]\,dt \tag{3.11}$$

where the path of integration is chosen as described above, then $\boldsymbol{\epsilon}_{jm,k}(z)$ also satisfies (3.10).

If λ_j is not an extreme eigenvalue, let $\mathfrak{A}(z)$ be holomorphic in a domain $\mathscr{D}$ which contains, or extends to infinity in, one of the sectors $\left|\arg z-(2k \pm \frac{1}{2})\frac{\pi}{r+1}\right| \leqq \frac{\pi-\alpha}{(r+1)}$ for some integer k in $0 \leqq k \leqq r$

[7] $\{\infty\} = \{e^{i\theta} \mid -\infty \leq \theta \leq \infty\}$.
[8] Compare [1], sec. 5.

and all z sufficiently large. We reorder, and if necessary relabel, the elements of the equation (3.10) so that we may write them in the form

$$\begin{aligned}\frac{d}{dz}\boldsymbol{\epsilon}_{jm}^{(1)}(z)-\mathbf{D}_j^{(1)}(z)\boldsymbol{\epsilon}_{jm}^{(1)}(z)&=z^{-2}\left[\alpha_1(z)\boldsymbol{\epsilon}_{jm}^{(1)}(z)+\alpha_2(z)\boldsymbol{\epsilon}_{jm}^{(2)}(z)\right]-\mathbf{R}_{jm}^{(1)}(z)\\ \frac{d}{dz}\boldsymbol{\epsilon}_{jm}^{(2)}(z)-\mathbf{D}_j^{(2)}(z)\boldsymbol{\epsilon}_{jm}^{(2)}(z)&=z^{-2}\left[\alpha_3(z)\boldsymbol{\epsilon}_{jm}^{(1)}(z)+\alpha_4(z)\boldsymbol{\epsilon}_{jm}^{(2)}(z)\right]-\mathbf{R}_{jm}^{(2)}(z)\end{aligned}\tag{3.12}$$

where the top line contains all rows of (3.10) such that $i\in\mathbf{N}_1$ and the bottom line contains all rows of (3.10) such that $i\in\mathbf{N}_2$ (see page 174). The diagonal matrix $\mathbf{D}_j(z)$ in (3.10) is accordingly subdivided into two diagonal matrices $\mathbf{D}_j^{(1)}(z)$ and $\mathbf{D}_j^{(2)}(z)$; the matrix $\mathfrak{B}(z)$ is accordingly partitioned into four blocks $\alpha_k(z)$, $(k=1, 2, 3, 4)$ and $\mathbf{R}_{jm}(z)$ is split into two vectors in the same manner as $\boldsymbol{\epsilon}_{jm}(z)$ was split.

Let $\mathscr{E}$ be the smallest closed disk with center at the origin containing the zeros of $q_{ij}'(t)$ $(i=1, 2, \ldots, n;\ i\neq j)$ and let $\mathscr{D}^*=\mathscr{D}-\mathscr{E}-\{\infty\}$. We next define a region $\mathscr{D}(\zeta_k^{(1)}, \zeta_k^{(2)})$ to be the union of all points z $(z\neq\zeta_k^{(1)}, z\neq\zeta_k^{(2)})$ such that there is a path $\mathscr{P}$ connecting $\zeta_k^{(1)}$, z and $\zeta_k^{(2)}$ (in that order) where $\zeta_k^{(1)}$ is one of the points $\infty\exp\left[(2k\pm1)\pi\sqrt{-1}/(r+1)\right]$, and $\zeta_k^{(2)}=\infty\exp\left[2\pi k\sqrt{-1}/(r+1)\right]$, and satisfying the following *interior eigenvalue conditions:*

(1) Except for $\zeta_k^{(1)}$ and $\zeta_k^{(2)}$, $\mathscr{P}$ lies entirely in $\mathscr{D}^*$;

(2) For any two distinct points t_1 and t_2 in the order $\zeta_k^{(1)}$, t_1, t_2, $\zeta_k^{(2)}$, on $\mathscr{P}$ we have $|\exp\left[q_{ij}(t_2)-q_{ij}(t_1)\right]|\leqq 1$ if $i\in\mathbf{N}_1$, $|\exp\left[q_{ij}(t_1)-q_{ij}(t_2)\right]|\leqq 1$ if $i\in\mathbf{N}_2$;

(3) With a_1, a_2, a_3, and a_4 defined by equation (3.32) below, $\frac{1}{2}\{a_1+a_4+\sqrt{(a_1-a_4)^2+4a_2a_3}\}$ $\mathscr{V}_{\mathscr{P}}(t^{-1})<1$.

It follows that the vector of analytic functions which satisfies

$$\begin{bmatrix}\boldsymbol{\epsilon}_{jm,k}^{(1)}(z)\\ \boldsymbol{\epsilon}_{jm,k}^{(2)}(z)\end{bmatrix}=\begin{Bmatrix}\int_{\zeta_k^{(1)}}^{z}e^{\mathbf{D}_j^{(1)}(z)-\mathbf{D}_j^{(1)}(t)} & \mathbf{0}\\ \mathbf{0} & \int_{\zeta_k^{(2)}}^{z}e^{\mathbf{D}_j^{(2)}(z)-\mathbf{D}_j^{(2)}(t)}\end{Bmatrix}\times\left\{t^{-2}\begin{bmatrix}\alpha_1(t) & \alpha_2(t)\\ \alpha_3(t) & \alpha_4(t)\end{bmatrix}\begin{bmatrix}\boldsymbol{\epsilon}_{jm,k}^{(1)}(t)\\ \boldsymbol{\epsilon}_{jm,k}^{(2)}(t)\end{bmatrix}-\begin{bmatrix}\mathbf{R}_{jm}^{(1)}(t)\\ \mathbf{R}_{jm}^{(2)}(t)\end{bmatrix}\right\}dt,\tag{3.13}$$

where the integrals are taken along a path $\mathscr{P}$ as described above, is a solution of (3.12).

3.3. Treatment of $\mathbf{R}_{i\,jm}(z)$

In this section we examine the integral

$$\mathbf{R}_{i\,jm}^{*}(z)=\int_{\zeta}^{z}e^{q_{ij}(z)-q_{ij}(t)}\mathbf{R}_{i\,jm}(t)dt\tag{3.14}$$

where the path $\mathscr{P}$ of integration satisfies either of the sets of conditions described in the previous section and ζ is a point appropriately chosen at infinity such that as t traverses $\mathscr{P}$ from ζ to z, Re $q_{ij}(t)$ decreases monotonically. The equations (3.4) and (3.6) show that when $i=j$, $\mathbf{R}_{i\,jm}(t)=\mathrm{O}(|t|^{-m-1})$, $|t|\to\infty$, but if $i\neq j$, then $\mathbf{R}_{ijm}(t)=\mathrm{O}(|t|^{r-m-1})$, $|t|\to\infty$. For this reason we cannot proceed directly as in [1] to bound the error.

When $i\neq j$ we first integrate (3.14) by parts to obtain a good bound for $\mathbf{R}_{i\,jm}^{*}(z)$. The equation (3.6) may be written

$$\mathbf{R}_{ijm}(z)=-e^{q_{ij}(z)}\frac{d}{dz}\left(e^{-q_{ij}(z)}\mathbf{U}_{ijm}z^{-m}\right)-z\left[q'_{ij}(z)-(q_{i0}-q_{j0})z^{r}+\frac{m}{z}\right]\mathbf{U}_{ijm}z^{-m-1}$$

$$-\sum_{\mu=m+1}^{m+r+1}\left[\sum_{k=\mu+1-m}^{\min(\mu,\,r+1)}(q_{ik}-q_{jk})\mathbf{U}_{ij,\,\mu-k}+(\mu-r-1)\mathbf{U}_{ij,\,\mu-r-1}\right.$$

$$\left.+\sum_{k=0}^{\mu-r-2}\{\mathbf{B}_k\mathbf{U}_{j,\,\mu-r-2-k}\}_i\right]z^{r-\mu}-\sum_{k=0}^{m-1}\{(\mathfrak{B}_{m-k}(z)\mathbf{U}_{jk})\}_i\,z^{-m-2}. \tag{3.15}$$

The first term on the right of (3.15) is already in a form suitable for integration by parts. Consider now the polynomial $\mathbf{P}_{ijm}(z)$ consisting of all of the right-hand side, excluding the first and last terms. It is clear that the dominant term of $\mathbf{P}_{ijm}(z)$ is of order z^{r-m-1} as $|z|\to\infty$. On substituting $\mathbf{P}_{ijm}(t)$ for $\mathbf{R}_{ijm}(t)$ in (3.14), integrating by parts and denoting the resulting integral by $\mathbf{P}^*_{ijm}$, we obtain

$$\mathbf{P}^*_{ijm}(z)=-\frac{\mathbf{P}_{ijm}(z)}{q'_{ij}(z)}+\int_{\zeta}^{z}e^{q_{ij}(z)-q_{ij}(t)}\frac{d}{dt}\left(\frac{\mathbf{P}_{ijm}(t)}{q'_{ij}(t)}\right)dt. \tag{3.16}$$

Collecting terms, we obtain

$$\mathbf{R}^*_{ijm}(z)=\int_{\zeta}^{z}\left[m\mathbf{U}_{ijm}t^{-m-1}+(e^{q_{ij}(z)-q_{ij}(t)}-1)\frac{d}{dt}\left(\frac{\mathbf{P}_{ijm}(t)}{q'_{ij}(t)}\right)-e^{q_{ij}(z)-q_{ij}(t)}\sum_{k=0}^{m-1}\{\mathfrak{B}_{m-k}(t)\mathbf{U}_{jk}\}_i t^{-m-2}\right]dt, \tag{3.17}$$

and since

$$|e^{q_{ij}(z)-q_{ij}(t)}|\leqq 1 \text{ and } \frac{d}{dt}\left(\frac{\mathbf{P}_{ijm}(t)}{q'_{ij}(t)}\right)=\mathrm{O}(|t|^{-m-2}),$$

$|t|\to\infty$, the right of (3.17) is readily bounded. In the next section, we use this bound in obtaining vector and norm bounds for $\boldsymbol{\epsilon}_{jm}(z)$.

On the other hand for $i=j$, we see from (3.4) that

$$\mathbf{R}^*_{jjm}(z)=\int_{\zeta}^{z}\left[m\mathbf{U}_{jjm}t^{-m-1}-\sum_{k=0}^{m-1}\{\mathfrak{B}_{m-k}(z)\mathbf{U}_{jk}\}_j t^{-m-2}\right]dt \tag{3.18}$$

and obtaining a bound for this integrand is even easier than for (3.17).

3.4. Error Bounds for the Extreme Eigenvalue Case

We shall first obtain a vector bound. With t_1 and t_2 in the order ζ_k, t_1, t_2, z on $\mathscr{P}$, we define

$$(m+1)\boldsymbol{\gamma}_{ijm}=\sup_{t_1,\,t_2\,\in\,\mathscr{P}}\left|e^{q_{ij}(t_2)-q_{ij}(t_1)}\left[t_1^{m+2}\frac{d}{dt}\left(\frac{\mathbf{P}_{ijm}(t)}{q'_{ij}(t)}\right)_{t=t_1}\right.\right.$$

$$\left.\left.+\sum_{k=1}^{m-1}\{\mathfrak{B}_{m-k}(t_1)\mathbf{U}_{jk}\}_i-\frac{d}{dt}\left(\frac{\mathbf{P}_{ijm}(t)}{q'_{ij}(t)}\right)_{t=t_1}t_1^{m+1}\right|\qquad(i\neq j)\right. \tag{3.19}$$

$$(m+1)\gamma_{jjm}=\sup_{t\,\in\,\mathscr{P}}\left|\sum_{k=0}^{m-1}\{\mathfrak{B}_{m-k}(t)\mathbf{U}_{jk}\}_j\right|.$$

We also denote by γ_{jm} the vector with ith element γ_{ijm}; i.e. $\{\gamma_{jm}\}_i = \gamma_{ijm}$, and define

$$\mathbf{B} = \left[\sup_{t_1,\, t_2 \in \mathscr{P}} \left| e^{q_{ij}(t_2) - q_{ij}(t_1)} \{\mathfrak{B}(t_1)\}_{is} \right| \right], \quad i, s = 1, 2, \ldots, n. \tag{3.20}$$

In the part II of the present paper we shall show that there exists a unique vector $\boldsymbol{\epsilon}_{jm,k}(z)$ of holomorphic functions which tends to zero as $z \to \zeta_k$ and satisfies (3.11). The elements of $\boldsymbol{\epsilon}_{jm,k}(z)$ are holomorphic in a domain (actually a Riemann surface) which includes the region $\mathscr{D}(z, \zeta_k)$ defined by the extreme eigenvalue conditions in Section 3.2.

On combining equations (3.19) and (3.20) with (3.11) we obtain

$$|\boldsymbol{\epsilon}_{jm,k}(z)| \leqq \int_{\zeta_k}^{z} [|t^{-2}\mathbf{B}\boldsymbol{\epsilon}_{jm,k}(t)| + m|\mathbf{U}_{jm}t^{-m-1}| + (m+1)\boldsymbol{\gamma}_{jm}t^{-m-2}]|dt|. \tag{3.21}$$

Let σ be the value of s for which $t(s) = z$ $t(s)$ as defined in the Introduction. We apply Lemmas 1 and 2 of the appendix with $E(\sigma) = 1$, $\mathbf{F}(s)ds = \mathbf{B}|t^{-2}dt|$, $\varphi(s) = |\boldsymbol{\epsilon}_{jm,k}(t)|$, $\mathbf{G}(s)ds = (m|\mathbf{U}_{jm}t^{-m-1}| + (m+1)\boldsymbol{\gamma}_{jm}|t^{-m-2}|)dt$ to obtain

THEOREM 3.2: *Corresponding to an extreme eigenvalue of λ_j of $\mathbf{C}_0$, the equation (2.2) possesses an actual solution vector $\mathbf{W}_{jm,k}(z)$ depending on ζ_k and an arbitrary positive integer m such that*

$$\mathbf{W}_{jm,k}(z) = \left[\sum_{s=0}^{m-1} \mathbf{U}_{js} z^{-s} + \boldsymbol{\epsilon}_{jm,k}(z) \right] e^{q_j(z)} \tag{3.22}$$

where

$$|\boldsymbol{\epsilon}_{jm,k}(z)| \leqq \exp\{\mathbf{B}\mathscr{V}_{\mathscr{P}}(t^{-1})\}\{|\mathbf{U}_{jm}|\mathscr{V}_{\mathscr{P}}(t^{-m}) + \boldsymbol{\gamma}_{jm}\mathscr{V}_{\mathscr{P}}(t^{-m-1})\}, \quad z \in \mathscr{D}(z, \zeta_k) \tag{3.23}$$

and $\mathscr{D}(z, \zeta_k)$ is defined in section 3.2. In (3.22) each vector $\mathbf{U}_{jk}$ ($k = 0, 1, 2, \ldots$) is the jth column vector of the matrix $\mathbf{U}_k$ defined by Theorem 2.2; $q_j(z)$ is the jth diagonal element of the diagonal matrix $\mathfrak{Q}(z)$ (equation (2.13)). In (3.23) $\mathbf{B}$ is an $n \times n$ matrix of non-negative elements defined by equations (2.3) and (3.20), while the ith element ($i = 1, 2, \ldots, n$) of the vector $\boldsymbol{\gamma}_{jm}$ is defined by equation (3.19).

Let us now obtain a norm bound for $\boldsymbol{\epsilon}_{jm,k}(z)$ in equation (3.22). For this purpose *we define a vector $\mathbf{V}_j(t_2, t_1)$ as follows: Let the ith element $(m+1)\mathbf{V}_{ij}(t_2, t_1)$ of $(m+1)\mathbf{V}_j(t_2, t_1)$ be the number inside the absolute value signs on the right of* (3.19). With t_1 and t_2 defined as for equation (3.19) we define[9]

$$B = \sup_{t \in \mathscr{P}} \|\mathfrak{B}(t)\|$$

$$\gamma_{jm} = \sup_{t_1,\, t_2 \in \mathscr{P}} \|\mathbf{V}_j(t_2, t_1)\|. \tag{3.24}$$

THEOREM 3.3: *A norm bound for the vector $\boldsymbol{\epsilon}_{jm,k}(z)$ in Theorem 3.2 is given by*

$$\|\boldsymbol{\epsilon}_{jm,k}(z)\| \leqq \exp\{B\mathscr{V}_{\mathscr{P}}(t^{-1})\}\{\|\mathbf{U}_{jm}\|\mathscr{V}_{\mathscr{P}}(t^{-m}) + \boldsymbol{\gamma}_{jm}\mathscr{V}_{\mathscr{P}}(t^{-m-1})\}. \tag{3.25}$$

[9] Here and henceforth we assume that compatible matrix and vector norms are taken, i.e., $\|\mathbf{BV}\| \leqq \|\mathbf{B}\|\,\|\mathbf{V}\|$.

3.5. Error Bounds for the Interior Eigenvalue Case

Put $t_2 = z$, $t_1 = t$, and let $\mathbf{V}_j(z, t)$ be defined as for (3.24). We split the vector $\mathbf{V}_j(z, t)$ into two vectors $\mathbf{V}_j^{(1)}(z, t)$ and $\mathbf{V}_j^{(2)}(z, t)$ in a manner corresponding to the way in which $\mathbf{R}_{jm}(t)$ was split in (3.13). We analogously split the vector $\mathbf{U}_{jm}$ into $\mathbf{U}_{jm}^{(1)}$ and $\mathbf{U}_{jm}^{(2)}$, such that the equation (3.13) may be written

$$\begin{bmatrix} \boldsymbol{\epsilon}_{jm,k}^{(1)}(z) \\ \boldsymbol{\epsilon}_{jm,k}^{(2)}(z) \end{bmatrix} = \begin{bmatrix} \int_{\zeta_k^{(1)}}^{z} & \mathbf{0} \\ \mathbf{0} & \int_{\zeta_k^{(2)}}^{z} \end{bmatrix} \left\{ \begin{bmatrix} e^{\mathbf{D}_j^{(1)}(z) - \mathbf{D}_j^{(1)}(t)} & \mathbf{0} \\ \mathbf{0} & e^{\mathbf{D}_j^{(2)}(z) - \mathbf{D}_j^{(2)}(t)} \end{bmatrix} t^{-2} \right.$$

$$\left. \times \begin{bmatrix} \mathfrak{a}_1(t) & \mathfrak{a}_2(t) \\ \mathfrak{a}_3(t) & \mathfrak{a}_4(t) \end{bmatrix} \begin{bmatrix} \boldsymbol{\epsilon}_{jm,k}^{(1)}(t) \\ \boldsymbol{\epsilon}_{jm,k}^{(2)}(t) \end{bmatrix} + \begin{bmatrix} -m\mathbf{U}_{jm}^{(1)} t^{-m-1} + (m+1)\mathbf{V}_j^{(1)}(z, t) t^{-m-2} \\ -m\mathbf{U}_{jm}^{(2)} t^{-m-1} + (m+1)\mathbf{V}_j^{(2)}(z, t) t^{-m-2} \end{bmatrix} \right\} dt. \qquad (3.26)$$

Again we shall first obtain a vector bound. To achieve this, we define

$$\mathbf{a}_k = \sup_{t \in \mathscr{P}} [\,|\{\mathfrak{a}_k(t)\}_{\text{is}}|\,] \qquad (k = 1, 2, 3, 4)$$

$$\mathbf{C}_m^{(s)}(t) = m|\mathbf{U}_{jm}^{(s)}||t^{-m-1}| + t_1,\ t_2 \overset{\sup}{\in} \mathscr{P} |(m+1)\mathbf{V}_{jm}^{(s)}(t_2, t_1)||t^{-m-2}| \qquad (3.27)$$

where in the last of (3.27) t_1 and t_2 are on $\mathscr{P}$ in the order ζ_1, t_1, t_2, ζ_2 for $s = 1$, and ζ_1, t_2, t_1, ζ_2 for $s = 2$ respectively.

In addition, each element of the diagonal matrices $e^{\mathbf{D}_j^{(s)}(z) - \mathbf{D}_j^{(s)}(t)}$ $(s = 1, 2)$ on the right of (3.23) has an upper bound of 1.

In part II of the present paper we shall show that there exists a unique $n \times 1$ vector of functions satisfying (3.26) such that $\boldsymbol{\epsilon}_{jm,k}^{(1)}(z) \to \mathbf{0}$ as $z \to \zeta_k^{(1)}$, $\boldsymbol{\epsilon}_{jm,k}^{(2)}(z) \to \mathbf{0}$ as $z \to \zeta_k^{(2)}$. Furthermore, each element of this vector is holomorphic on a Riemann surface which includes the region $\mathscr{D}(\zeta_k^{(1)}, \zeta_k^{(2)})$ defined by the interior eigenvalue conditions of section 3.2.

On substituting the above results in (3.26), we obtain

$$\begin{bmatrix} |\boldsymbol{\epsilon}_{jm,k}^{(1)}(z)| \\ |\boldsymbol{\epsilon}_{jm,k}^{(2)}(z)| \end{bmatrix} \leqq \begin{bmatrix} \int_{\zeta_k^{(1)}}^{z} & \mathbf{0} \\ \mathbf{0} & \int_{\zeta_k^{(2)}}^{z} \end{bmatrix} \left\{ \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \\ \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} \begin{bmatrix} |\boldsymbol{\epsilon}_{jm,k}^{(1)}(t)| \\ |\boldsymbol{\epsilon}_{jm,k}^{(2)}(t)| \end{bmatrix} |t^{-2}| + \begin{bmatrix} \mathbf{C}_m^{(1)}(t) \\ \mathbf{C}_m^{(2)}(t) \end{bmatrix} \right\} |dt|. \qquad (3.28)$$

On transforming to real variables as for (3.21), using Lemma 5 of the appendix and transforming back to complex variable notation, we obtain

THEOREM 3.3: *If, for an interior eigenvalue* λ_j *of* $\mathbf{C}_0$, *the interior eigenvalue conditions of section 3.2 are satisfied, then the equation* (2.2) *possesses an actual solution vector*

$$\begin{Bmatrix} \mathbf{W}_{jm,k}^{(1)}(t) \\ \mathbf{W}_{jm,k}^{(2)}(z) \end{Bmatrix} = \left\{ \sum_{s=0}^{m-1} \begin{bmatrix} \mathbf{U}_{js}^{(1)} \\ \mathbf{U}_{js}^{(2)} \end{bmatrix} z^{-s} + \begin{bmatrix} \boldsymbol{\epsilon}_{jm,k}^{(1)}(z) \\ \boldsymbol{\epsilon}_{jm,k}^{(2)}(z) \end{bmatrix} \right\} e^{\phi_j^{(s)}} \qquad (3.29)$$

where

$$|\boldsymbol{\epsilon}_{jm,k}^{(1)}(z)| \leqq \exp\{\mathbf{a}_1 \mathscr{V}_{\zeta_k^{(1)}, z}(t^{-1})\} \psi_m^{(1)}(\zeta_k^{(1)}, z) + \mathbf{C}_1(\zeta_k^{(1)}, z)[\exp\{\mathbf{a}_4 \mathscr{V}_{\mathscr{P}}(t^{-1})\} \psi_m^{(2)}(\zeta_k^{(1)}, \zeta_k^{(2)}) + \mathbf{B}],\ (z \in \mathscr{D}(\zeta_k^{(1)}, \zeta_k^{(2)}))$$

and

$$\psi_m^{(s)}(u, v) = \int_u^v \mathbf{C}_m^{(s)}(t)|dt| \qquad (s = 1, 2) \tag{3.30}$$

$$\mathbf{B} = [\mathbf{I}_2 - \mathbf{C}_2\mathbf{C}_1(\zeta_k^{(1)}, \zeta_k^{(2)})]^{-1}\mathbf{C}_2[\exp\{\mathbf{a}_1\mathscr{V}_{\mathscr{P}}(t^{-1})\}\psi_m^{(1)}(\zeta_k^{(1)}, \zeta_k^{(2)}) + \mathbf{C}_1(\zeta_k^{(1)}, \zeta_k^{(2)}) \exp\{\mathbf{a}_4\mathscr{V}_{\mathscr{P}}(t^{-1})\}\psi_m^{(2)}(\zeta_k^{(1)}, \zeta_k^{(2)})]$$

$$\mathbf{C}_1(\zeta_k^{(1)}, z) = \mathbf{a}_1^{-1}[\exp\{\mathbf{a}_1\mathscr{V}_{\mathscr{P}}(t^{-1})\} - \mathbf{I}_1]\mathbf{a}_2;\ \mathbf{C}_2 = \mathbf{a}_4^{-1}[\exp\{\mathbf{a}_4\mathscr{V}_{\mathscr{P}}(t^{-1})\} - \mathbf{I}_2].$$

Here $\mathbf{I}_1$ *designates the* $\kappa \times \kappa$ *unit matrix, and* $\mathbf{I}_2$ *designates the* $(n-\kappa)\times(n-\kappa)$ *unit matrix,* κ *being the number of elements in* $\mathbf{N}_1$ (Section 3.2). *All integrals and variations on the right of* (3.30) *are taken along* $\mathscr{P}$. *The bound on the right of* (3.30) *is valid provided that every eigenvalue of the matrix* $\mathbf{C}_2\mathbf{C}_1(\zeta_k^{(1)}, \zeta_k^{(2)})$ *is less than* 1 *in magnitude. Moreover, an exactly similar result holds for* $\epsilon_{jm,k}^{(2)}(z)$.

Let us now obtain a norm bound. We define

$$\psi_m^{(s)}(u, v) = \|\mathbf{U}_{jm}^{(s)}\|\mathscr{V}_{u,v}(t^{-m}) + \sup_{t_1, t_2 \in \mathscr{P}} \|\mathbf{V}_j^{(s)}(t_2, t_1)\|\mathscr{V}_{u,v}(t^{-m-1}) \tag{3.31}$$

where t_1 and t_2 are any two points on $\mathscr{P}$ in the order $\zeta_k^{(1)}, t_1, t_2, \zeta_k^{(2)}$ for $s = 1$ and in the order $\zeta_k^{(1)}, t_2, t_1, \zeta_k^{(2)}$ for $s = 2$. Similarly, with the notation of (3.27) we put

$$a_k = \sup_{t \in \mathscr{P}} \|\alpha_k(t)\| \qquad (k = 1, 2, 3, 4). \tag{3.32}$$

Thus we obtain an inequality similar to (3.29); using Lemma 6 and the inequality (32) of the appendix we then obtain

THEOREM 3.4: *Let* a_k $(k = 1, 2, 3, 4)$ *be defined by* (3.32), *let* $\psi_m^{(s)}$ $(k = 1, 2)$ *be defined by* (3.31), *and let*

$$\begin{aligned} f(z) &= \mathscr{V}_{\zeta_k^{(1)}, z}(t^{-1}) \\ F(z) &= \frac{a_2a_3}{(a_1 + a_4)^2}\,[e^{(a_1+a_4)f(z)} - 1 - (a_1 + a_4)f(z)] \end{aligned} \tag{3.33}$$

where the variations are taken along $\mathscr{P}$. *If* $F(\zeta_k^{(2)}) < 1$ *then the norm of the vector* $\epsilon_{jm,k}^{(1)}(z)$ (equation (3.29)) *is bounded by the quantity*

$$\begin{aligned} &(1 - F(\zeta_k^{(1)}))^{-1}\{e^{a_1f(z)}[F(z) + 1 - F(\zeta_k^{(2)})]\psi_m(\zeta_k^{(1)}, z) \\ &\quad + \frac{a_2a_3}{(a_1 + a_4)^2}\,[1 - e^{-(a_1+a_4)f(z)}]\,[e^{(a_1+a_4)f(\zeta_k^{(2)})} - e^{(a_1+a_4)f(z)}]\psi_m^{(1)}(z, \zeta_k^{(2)}) \\ &\quad + \frac{a_2}{a_1 + a_4}\,e^{a_4f(z) + a_4f(\zeta_k^{(2)})}[1 - e^{-(a_1+a_4)f(z)}]\psi_m^{(2)}(\zeta_k^{(1)}, \zeta_k^{(2)}). \end{aligned} \tag{3.34}$$

Moreover, an exactly similar result holds for $\epsilon_{jm,k}^{(2)}(z)$.

It is noteworthy that the conditions of Theorems 3.3 and 3.4 can always be satisfied by taking contours $\mathscr{P}$ sufficiently far away from the origin.

4. Appendix. Some Inequalities for Volterra and Fredholm Integral Equations

4.1. Bellman's Lemma in Matrix Form

We begin with an extension of Bellman's Lemma [5] to matrices.

LEMMA 1: *Let* φ, $\mathbf{G}$ *and* $\mathbf{H}$ *be* $k \times 1$ *vectors and* $\mathbf{E}$ *and* $\mathbf{F}$ *be* $k \times k$ *matrices of continuous non-negative functions of a variable* σ *in an interval* $a \leqq \sigma \leqq b$ *such that* $\mathbf{F}(\sigma)\,\mathbf{E}(\sigma)$ *and* $\int^{\sigma} \mathbf{F}(s)\mathbf{E}(s)ds$ *commute. If*[10]

$$\varphi(\sigma) \leqq \mathbf{E}(\sigma) \int_a^{\sigma} [\mathbf{F}(s)\varphi(s) + \mathbf{G}(s)]ds + \mathbf{H}(\sigma) \qquad (a \leqq \sigma \leqq b) \tag{1}$$

then the following inequality also holds in $a \leqq \sigma \leqq b$:

$$\varphi(\sigma) \leqq \mathbf{H}(\sigma) + \mathbf{E}(\sigma) \int_a^{\sigma} \exp\left\{\int_s^{\sigma} \mathbf{F}(\tau)\mathbf{E}(\tau)d\tau\right\} \{\mathbf{G}(s) + \mathbf{F}(s)\mathbf{H}(s)\}ds. \tag{2}$$

The proof of Lemma 1 is a straightforward extension of the one-dimensional result in [5, page 135].

LEMMA 2: *With the conditions of Lemma 1, let* $\mathbf{E}^{-1}(\sigma)$ and $\mathscr{V}_{a,b}(\mathbf{E}^{-1}\mathbf{H})$ *exist for* σ *in* (a, b). *Then*

$$\varphi(\sigma) \leqq \mathbf{E}(\sigma) \exp\left\{\int_a^{\sigma} \mathbf{F}(\tau)\mathbf{E}(\tau)d\tau\right\} \left\{\mathbf{E}^{-1}(a)\mathbf{H}(a) + \mathscr{V}_{a,\sigma}(\mathbf{E}^{-1}\mathbf{H}) + \int_a^{\sigma} \mathbf{G}(s)ds\right\}. \tag{3}$$

To prove this result we integrate the right of (2) by parts.

If $\mathbf{E}^{-1}(s)\mathbf{H}(s)$ increases monotonically over (a,b), then $\mathscr{V}_{a,\sigma}(\mathbf{E}^{-1}\mathbf{H}) = \mathbf{E}^{-1}(\sigma)\mathbf{H}(\sigma) - \mathbf{E}^{-1}(a)\mathbf{H}(a)$; in this case the inequality (3) may be simplified. If, in addition, $\mathbf{E}(s)$ is a scalar quantity then (3) reduces to

$$\varphi(\sigma) \leqq \exp\left\{\int_a^{\sigma} \mathbf{F}(\tau)\mathbf{E}(\tau)d\tau\right\} \left\{\mathbf{H}(\sigma) + \mathbf{E}(\sigma) \int_a^{\sigma} \mathbf{G}(s)ds\right\}. \tag{4}$$

4.2. Bellman's Lemma for Fredholm Vector Integral Equations

The following lemma extends the above results to Fredholm integral equations.

LEMMA 3: *When the variables* σ, τ, *and* t *lie in the interval* (a, b), *let* $\mathbf{X}(\sigma, \tau)$ $(k \times k)$, $\varphi(\tau)$ $(k \times 1)$, *and* $\theta(\tau)$ $(k \times 1)$ *be matrices of non-negative continuous functions such that*

$$\mathbf{X}(\sigma, \tau)\,\mathbf{X}(\tau, t) \leqq \mathbf{X}(\sigma, t)\,\mathbf{X}(\tau, \tau) \tag{5}$$

and

$$\varphi(\sigma) \leqq \int_a^b \mathbf{X}(\sigma, \tau)\varphi(\tau)d\tau + \theta(\sigma). \tag{6}$$

Suppose that each eigenvalue of the matrix

$$\mathbf{F} = \int_a^b \mathbf{X}(\tau, \tau)d\tau \tag{7}$$

is less than 1 *in magnitude. Then*

$$\varphi(\sigma) \leqq \theta(\sigma) + \int_a^b \mathbf{X}(\sigma, \tau)(\mathbf{I} - \mathbf{F})^{-1}\theta(\tau)d\tau \quad (a \leqq \sigma \leqq b). \tag{8}$$

---

[10] If (a, b) is not compact we assume that the integral on the right of (1) exists and that each element of the resulting vector is uniformly bounded.

PROOF: From the conditions of the Lemma there exists a unique $k \times 1$ vector $\Psi(\sigma)$ of non-negative continuous functions such that we can replace the inequality (6) by the equality

$$\varphi(\sigma) = \int_a^b \mathbf{X}(\sigma, \tau)\varphi(\tau)d\tau + \theta(\sigma) - \Psi(\sigma). \tag{9}$$

The equation (9) is a Fredholm integral equation in φ which we can solve by successive approximations. Defining a sequence $\{\varphi_\nu\}(\nu = 0, 1, \ldots)$ by $\varphi_0 = \mathbf{O}$,

$$\varphi_{\nu+1}(\sigma) = \int_a^b \mathbf{X}(\sigma, \tau)\varphi_\nu(\tau)d\tau + \theta(\sigma) - \Psi(\sigma), \qquad \nu = 0, 1, \ldots, \tag{10}$$

and using (5), we easily establish by induction that

$$\mathbf{O} \leqq \varphi_{\nu+1}(\sigma) - \varphi_\nu(\sigma) \leqq \int_a^b \mathbf{X}(\sigma, \tau)\mathbf{F}^{\nu-1}(\theta(\tau) - \Psi(\tau))d\tau, \qquad \nu = 1, 2, \ldots. \tag{11}$$

If the eigenvalues of $\mathbf{F}$ are all less than 1 in magnitude, we have

$$(\mathbf{I} - \mathbf{F})^{-1} = \sum_{\nu=0}^{\infty} \mathbf{F}^\nu \tag{12}$$

and clearly the sum of the power series on the right is a matrix of non-negative elements. On summing the inequalities (11), we obtain (8) with θ replaced by $\theta - \Psi$, since it is well known that this sum bounds the true solution of (9). Decreasing the magnitude of the elements of Ψ on the right of this sum increases the elements of φ. The inequality (8) now follows.

The proof of the following Lemma is similar to that of the above Lemma, and is omitted.

LEMMA 4: *With the conditions of Lemma 3 let* (5) *be replaced by*

$$\mathbf{X}(\sigma, \tau) \leqq \mathbf{G}(\sigma)\mathbf{H}(\tau) \tag{5'}$$

where $\mathbf{G}$ *and* $\mathbf{H}$ *are* $\mathrm{k} \times \mathrm{k}$ *matrices. Let each eigenvalue of*

$$\mathbf{F} = \int_{\mathrm{a}}^{\mathrm{b}} \mathbf{H}(\tau)\mathbf{G}(\tau)\mathrm{d}\tau \tag{7'}$$

be less than 1 *in magnitude. Then*

$$\varphi(\sigma) \leqq \theta(\sigma) + \int_{\mathrm{a}}^{\mathrm{b}} \mathbf{G}(\sigma)(\mathbf{I} - \mathbf{F})^{-1}\mathbf{H}(\tau)\theta(\tau)\mathrm{d}\tau \qquad (\mathrm{a} \leqq \sigma \leqq \mathrm{b}). \tag{8'}$$

4.3. Extensions to a Simultaneous Pair of Volterra Vector Integral Equations

Here we consider the pair of inequalities

$$\begin{aligned} \varphi_1(\sigma) &\leqq \int_a^\sigma \{[\boldsymbol{\alpha}\varphi_1(s) + \boldsymbol{\beta}\varphi_2(s)]\omega(s) + \theta_1(s)\}ds \\ \varphi_2(\sigma) &\leqq \int_\sigma^b \{[\boldsymbol{\gamma}\varphi_1(s) + \boldsymbol{\delta}\varphi_2(s)]\omega(s) + \theta_2(s)\}ds \end{aligned} \tag{13}$$

where φ_1 $(\kappa \times 1)$, φ_2 $((n-\kappa) \times 1)$, θ_1 $(\kappa \times 1)$, θ_2 $((n-\kappa) \times 1)$, ω (1×1) are matrices of nonnegative continuous functions on (a, b), and $n \geqq 2$, $0 < \kappa < n$. Also, $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$, and $\boldsymbol{\delta}$ are constant matrices of orders $\kappa \times \kappa$, $\kappa \times (n-\kappa)$, $(n-\kappa) \times \kappa$ and $(n-\kappa) \times (n-\kappa)$ respectively, whose elements are nonnegative.

On applying Lemma 1 to each of the equations (13), we obtain

$$\begin{aligned}\varphi_1(\sigma) &\leqq \int_a^{\sigma} e^{\alpha f(\sigma)-\alpha f(s)}\{\beta\varphi_2(s)\omega(s)+\theta_1(s)\}\,ds \\ \varphi_2(\sigma) &\leqq \int_{\sigma}^{b} e^{\delta f(s)-\delta f(\sigma)}\{\gamma\varphi_1(s)\omega(s)+\theta_2(s)\}\,ds\end{aligned} \tag{14}$$

where

$$f(s)=\int_a^s \omega(\tau)\,d\tau. \tag{15}$$

Substituting the second of (14) for φ_2 in the first of (14), we get

$$\varphi_1(\sigma) \leqq \int_a^{\sigma}\int_s^{b} e^{\alpha f(\sigma)-\alpha f(s)}\,[\beta e^{\delta f(\tau)-\delta f(s)}\,\{\gamma\varphi_1(\tau)\omega(\tau)+\theta_2(\tau)\}d\tau\omega(s)+\theta_1(s)]ds. \tag{16}$$

The repeated integral (16) may be interpreted as a double integral over the shaded region indicated in figure 3. Splitting this region into a triangle and a rectangle and interchanging the order of integration, we obtain

$$\begin{aligned}\varphi_1(\sigma) \leqq \theta_3(\sigma)+\int_a^{\sigma}\int_a^{\tau} e^{\alpha f(\sigma)-\alpha f(s)}\,\beta e^{\delta f(\tau)-\delta f(s)}\,\}\gamma\varphi_1(\tau)\omega(\tau)+\theta_2(t)\}\omega(s)ds\,d\tau \\ +\int_{\sigma}^{b}\int_a^{\sigma} e^{\alpha f(\sigma)-\alpha f(s)}\,\beta e^{\delta f(\tau)-\delta f(s)}\,\{\gamma\varphi_1(\tau)\omega(\tau)+\theta_2(\tau)\}\omega(s)ds\,d\tau,\end{aligned} \tag{17}$$

where

$$\theta_3(\sigma)=\int_a^{\sigma} e^{\alpha f(\sigma)-\alpha f(s)}\,\theta_1(s)ds. \tag{18}$$

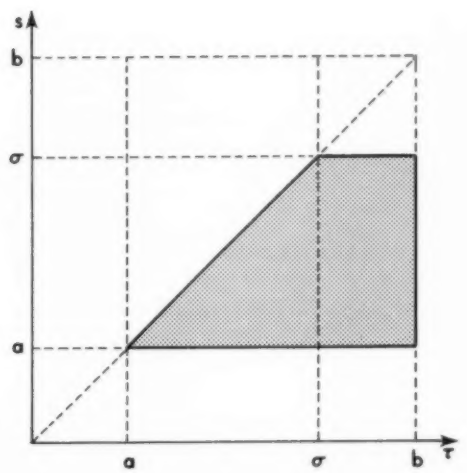

FIGURE 3.

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We evaluate the inner integrals in (17) to find that

$$\varphi_1(\sigma) \leqq \theta_3(\sigma) + \int_a^b \mathbf{X}(\sigma, \tau)\{\boldsymbol{\gamma}\varphi_1(\tau)\omega(\tau) + \theta_2(\tau)\}\, d\tau \tag{19}$$

where $\mathbf{X}(\sigma, \tau)$ satisfies

$$\boldsymbol{\alpha}\mathbf{X}(\sigma, \tau) + \mathbf{X}(\sigma, \tau)\boldsymbol{\delta} = \begin{cases} e^{\boldsymbol{\alpha} f(\sigma)}[\boldsymbol{\beta} e^{\boldsymbol{\delta} f(\tau)} - e^{-\boldsymbol{\alpha} f(\tau)}\boldsymbol{\beta}], & \tau \leqq \sigma \\ [e^{\boldsymbol{\alpha} f(\sigma)}\boldsymbol{\beta} - \boldsymbol{\beta} e^{-\boldsymbol{\delta} f(\sigma)}]e^{\boldsymbol{\delta} f(\tau)}, & \tau \geqq \sigma. \end{cases} \tag{20}$$

The equations (20) have unique solutions for $\mathbf{X}(\sigma, \tau)$ provided that $\mu_i + \nu_j \neq 0$ ($i = 1, 2, \ldots, \kappa$; $j = 1, 2, \ldots, n - \kappa$), where μ_i and ν_j are the eigenvalues of $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$, respectively. Moreover, this condition can always be satisfied by an arbitrarily small change in the elements of $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$, if necessary. Nevertheless an exact explicit expression for the right hand side of (20) is generally not feasible, and in practice we make a slight sacrifice of sharpness of error bound in order to achieve a matrix bound which is simpler to evaluate. Several simplifications are possible; one of these is as follows:[11]

$$\int_a^\tau e^{\boldsymbol{\alpha} f(\sigma) - \boldsymbol{\alpha} f(s)}\, \boldsymbol{\beta} e^{\boldsymbol{\delta} f(\tau) - \boldsymbol{\delta} f(s)}\, \omega(s) ds \leqq \int_a^\tau e^{\boldsymbol{\alpha} f(\sigma) - \boldsymbol{\alpha} f(s)}\, \boldsymbol{\beta} e^{\boldsymbol{\delta} f(\tau)}\, \omega(s) ds$$
$$= \boldsymbol{\alpha}^{-1}[e^{\boldsymbol{\alpha} f(\sigma)} - e^{\boldsymbol{\alpha} f(\sigma) - \boldsymbol{\alpha} f(\tau)}]\boldsymbol{\beta} e^{\boldsymbol{\delta} f(\tau)} \leqq \boldsymbol{\alpha}^{-1}[e^{\boldsymbol{\alpha} f(\sigma)} - \mathbf{I}_\kappa]\boldsymbol{\beta} e^{\boldsymbol{\delta} f(\tau)}. \tag{21}$$

Similarly

$$\int_a^\sigma e^{\boldsymbol{\alpha} f(\sigma) - \boldsymbol{\alpha} f(s)} \boldsymbol{\beta} e^{\boldsymbol{\delta} f(\tau) - \boldsymbol{\delta} f(s)}\, \omega(s) ds \leqq \boldsymbol{\alpha}^{-1}[e^{\boldsymbol{\alpha} f(\sigma)} - \mathbf{I}_\kappa]\boldsymbol{\beta} e^{\boldsymbol{\delta} f(\tau)} \equiv \mathbf{G}(\sigma)\mathbf{H}(\tau), \tag{22}$$

where

$$\mathbf{G}(\sigma) = \boldsymbol{\alpha}^{-1}[e^{\boldsymbol{\alpha} f(\sigma)} - \mathbf{I}_\kappa], \qquad \mathbf{H}(\tau) = \boldsymbol{\beta} e^{\boldsymbol{\delta} f(\tau)}. \tag{23}$$

Combining (22) and (23), we obtain

$$\varphi_1(\sigma) \leqq \theta_3(\sigma) + \int_a^b \mathbf{G}(\sigma)\mathbf{H}(\tau)\{\boldsymbol{\gamma}\varphi_1(\tau)\omega(\tau) + \theta_2(\tau)\}\, d\tau. \tag{24}$$

We apply Lemma 4 to this inequality to obtain:

LEMMA 5: *Let* φ_1, φ_2, α, β, γ, δ, ω, θ, *and* θ_2 *be defined as in* (13) *and the functions* $\mathbf{G}$ *and* $\mathbf{H}$ *as in* (23). *If the inequalities* (13) *hold and if each eigenvalue of the matrix*

$$\mathbf{F} = \int_a^b \mathbf{H}(\tau)\boldsymbol{\gamma}\mathbf{G}(\tau)\omega(\tau) d\tau \tag{25}$$

is less than 1 *in magnitude, then*

$$\varphi_1(\sigma) \leqq \theta(\sigma) + \int_a^b \mathbf{G}(\sigma)(\mathbf{I} - \mathbf{F})^{-1}\mathbf{H}(s)\boldsymbol{\gamma}\theta(s)\omega(s) ds, \tag{26}$$

[11] Since $\boldsymbol{\alpha}^{-1}[\exp(\boldsymbol{\alpha}) - \mathbf{I}] = \sum_{k=0}^{\infty} \frac{\boldsymbol{\alpha}^k}{(k+1)!}$ the possibility of $\boldsymbol{\alpha}$ being singular is not excluded.

where

$$\theta(\sigma)=\theta_3(\sigma)+\int_a^b \mathbf{G}(\sigma)\mathbf{H}(\tau)\theta_2(\tau)d\tau \tag{27}$$

and θ_3 *is defined by* (18). *Moreover an exactly similar result holds for* $\varphi_2(\sigma)$.

In the case when $n=2$ and $\kappa=1$ the equations (20) can be solved explicitly. We then have (19) with

$$\mathbf{X}(\sigma,\tau)=\begin{cases}\dfrac{\boldsymbol{\beta}}{\boldsymbol{\alpha}+\boldsymbol{\delta}}\,e^{\alpha f(\sigma)}[e^{\delta f(\tau)}-e^{-\alpha f(\tau)}], & \tau\leqq\sigma;\\ \dfrac{\boldsymbol{\beta}}{\boldsymbol{\alpha}+\boldsymbol{\delta}}\,e^{\delta f(\tau)}[e^{\alpha f(\sigma)}-e^{-\delta f(\sigma)}], & \tau\geqq\sigma.\end{cases} \tag{28}$$

It is easily verified that for arbitrary points σ, τ and t on (a,b), $\mathbf{X}(\sigma,\tau)\mathbf{X}(\tau,t)\leqq\mathbf{X}(\sigma,t)\mathbf{X}(\tau,\tau)$; i.e., the conditions of Lemma 3 are satisfied. On applying Lemma 3 to the inequality (19) we obtain

LEMMA 6: *Let* φ_1, φ_2, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$, ω, θ_1 *and* θ_2 *be defined as in* (13), *with* $n=2$, $\kappa=1$, *and*

$$\mathbf{F}(\sigma)=\frac{\boldsymbol{\beta}\boldsymbol{\gamma}}{(\boldsymbol{\alpha}+\boldsymbol{\delta})^2}\,[e^{(\alpha+\delta)f(\sigma)}-1-(\boldsymbol{\alpha}+\boldsymbol{\delta})f(\sigma)], \tag{29}$$

where f(s) *is defined by* (15). *If the inequalities* (13) *hold and if* $\mathrm{F(b)}<1$, *then*

$$\varphi_1(\sigma)\leqq\theta_3(\sigma)+\frac{1}{1-\mathbf{F}(\mathrm{b})}\int_a^b \mathbf{X}(\sigma,\tau)[\boldsymbol{\gamma}\omega(\tau)\theta_3(\tau)+\theta_2(\tau)]d\tau \qquad (\mathrm{a}\leqq\sigma\leqq\mathrm{b}) \tag{30}$$

where $\mathbf{X}(\sigma,\tau)$ *is defined by* (28) *and* θ_3 *is defined by* (18).

On expanding the right hand side of (30) and using the inequality $\mathbf{X}(\sigma,\tau)\mathbf{X}(\tau,t)\leqq\mathbf{X}(\sigma,t)\mathbf{X}(\tau,\tau)$ once more, we get

$$\begin{aligned}\varphi_1(\sigma)\leqq\frac{\boldsymbol{\beta}e^{\alpha f(\sigma)}}{1-\mathbf{F}(b)(\boldsymbol{\alpha}+\boldsymbol{\delta})}\Bigg[\int_a^\sigma\left\{\frac{\boldsymbol{\alpha}+\boldsymbol{\delta}}{\boldsymbol{\beta}}\,[\mathbf{F}(\sigma)-\mathbf{F}(s)+1-\mathbf{F}(b)]e^{-\alpha f(s)}\theta_1(s)+[e^{\delta f(s)}-e^{-\alpha f(s)}]\theta_2(s)\right\}ds\\ +[1-e^{-(\alpha+\delta)f(\sigma)}]\int_\sigma^b\left\{\frac{\boldsymbol{\gamma}}{\boldsymbol{\alpha}+\boldsymbol{\delta}}\,[e^{(\alpha+\delta)f(b)}-e^{(\alpha+\delta)f(s)}]e^{-\alpha f(s)}\theta_1(s)+e^{\delta f(s)}\theta_2(s)\right\}ds\Bigg].\end{aligned} \tag{31}$$

For ease of evaluation we make some over-estimates in (31) to obtain

$$\begin{aligned}\varphi_1(\sigma)\leqq\frac{1}{1-\mathbf{F}(b)}\Bigg\{e^{\alpha f(\sigma)}[\mathbf{F}(\sigma)+1-\mathbf{F}(b)]\int_a^\sigma\theta_1(\tau)d\tau\\ +\frac{\boldsymbol{\beta}\boldsymbol{\gamma}}{(\boldsymbol{\alpha}+\boldsymbol{\delta})^2}\,[1-e^{-(\alpha+\delta)f(\sigma)}][e^{(\alpha+\delta)f(b)}-e^{(\alpha+\delta)f(\sigma)}]\int_\sigma^b\theta_1(\tau)d\tau\\ +\frac{\boldsymbol{\beta}}{\boldsymbol{\alpha}+\boldsymbol{\delta}}\,e^{\alpha f(\sigma)+\delta f(b)}[1-e^{-(\alpha+\delta)f(\sigma)}]\int_a^b\theta_2(\tau)d\tau\Bigg\}.\end{aligned} \tag{32}$$

In conclusion, we observe that if we replace all inequalities in Lemmas 1, 2, 3, and 4 by equalities, then the results of these lemmas are exact; hence the inequalities are sharp. However, since $\mathbf{X}(\sigma,\tau)<\mathbf{G}(\sigma)\mathbf{H}(\tau)$ almost everywhere in $a\leqq\sigma,\ \tau\leqq b$ the result of Lemma 5 is not sharp. The

result of Lemma 6 also is not sharp because, although $\mathbf{X}(\sigma, \tau)\mathbf{X}(\tau, t) = \mathbf{X}(\sigma, t)\mathbf{X}(\tau, \tau)$ for σ, τ, t in the order σ, τ, t or t, τ, σ on (a,b), $\mathbf{X}(\sigma, \tau)\mathbf{X}(\tau, t) < \mathbf{X}(\sigma, t)\mathbf{X}(\tau, \tau)$ when this order of σ, τ and t is violated.

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5. References

[1] Olver, F. W. J., On the asymptotic solution of second order differential equations having an irregular singularity of rank one, with an application to Whittaker functions, J. Soc. Indust. Appl. Math. Numer. Anal. Ser. B **2,** 225–243 (1965).
[2] Trjitzinsky, W. J., Analytic theory of linear differential equations, Acta. Math. **62,** 167–226 (1933).
[3] Birkhoff, G. D., Singular points of ordinary differential equations, Trans. Amer. Math. Soc. **10,** 436–470 (1909).
[4] Olver, F. W. J., On error bounds for asymptotic expansions, with application to Bessel Functions of large argument; Asymptotic Solutions of Differential Equations and their Applications, C. H. Wilcox, ed., John Wiley, New York (1964).
[5] Beckenbach, E. F., and Bellman, R., Inequalities, Springer-Verlag (1961).
[6] Olver, F. W. J., and Stenger, F., Error bounds for asymptotic solutions of second-order differential equations having an irregular singularity of arbitrary rank, J. Soc. Indust. Appl. Math. Num. Anal. Ser. B **2,** 244–249 (1965).

(Paper 70B3–180)

Error Bounds for Asymptotic Solutions of Differential Equations. II. The General Case[1]

Frank Stenger[2]

(May 26, 1966)

The results of the preceding paper are extended to the general system of n first-order differential equations having an irregular singularity of arbitrary rank at infinity. Formal solutions are explicitly constructed for the system in canonical form. Proofs of existence and uniqueness of solutions of integral equations defining the error are given. As an example, the case $n=2$ is solved completely, and a flow chart of the transformations of this case to canonical form is included.

1. Introduction

The system of differential equations under consideration in this paper is of the form

$$\frac{d\mathbf{W}}{dz}=z^h\left[\sum_{k=0}^{s-1}\mathbf{A}_k z^{-k}+\mathfrak{R}_s(z)z^{-s}\right]\mathbf{W}, \qquad s=0, 1, \ldots, \tag{1.1}$$

where h is an integer and the matrices $\mathbf{A}_k$ and $\mathfrak{R}_s(z)$ are defined as in [1], except that no assumptions are made concerning eigenvalues of the matrices $\mathbf{A}_k$ $(k=0, 1, \ldots)$. We shall simultaneously consider the case of a regular point, the case of a regular singular point, and the case of an irregular singular point at infinity.

From the point of view of applications it is worthwhile recalling that when the eigenvalues of the lead coefficient matrix $\mathbf{A}_0$ are not distinct they can be made distinct by an arbitrarily small change in the elements of $\mathbf{A}_0$. The results of [1][3] can then be applied, and the very complicated procedure given in this paper for obtaining a formal solution of (1.1) avoided. Indeed the existence of a linearly independent set of n formal solution vectors for the system (1.1) was fully established only in 1937 ([2]) and more explicitly in 1955 ([3]).

We have not been able to improve on the method of [3] for the reduction of (1.1) to canonical form. We thus start with the canonical form given in [3] and establish a minor variation of it which enables us more easily to obtain error bounds. Next we explicitly construct formal solutions for the system in canonical form. Integral equations are then set up for the error vector, that is, for the difference between the partial sum of a formal solution vector and an actual solution vector.

As in [1], it is again possible to express the error by one or, at most, two Volterra vector integral equations. As opposed to [1], however, it is not always possible to choose the end points of integration in the Volterra integral equations to be at infinity.

2. The Canonical Form

The following theorem due to H. L. Turrittin [3] concerns transformation to canonical form, of the system (1.1), with eigenvalues of $\mathbf{A}_0$ not necessarily distinct. In this theorem the normalizing, exponential, zero-inducing, root-equalizing, and shearing transformations are defined as in [3].

[1] An invited paper based on the author's Ph. D thesis, University of Alberta (1965).
[2] Present address: University of Michigan, Ann Arbor, Mich.
[3] Figures in brackets indicate the literature references at the end of this paper.

An example of each of these transformations will be given in our treatment of the special $n=2$ case in section 4.

THEOREM 2.1: *Let h be an integer and*

$$\frac{d\mathbf{X}}{dt} = t^h \mathfrak{A}(t)\mathbf{X} \tag{2.1}$$

a system of differential equations in which

$$\mathfrak{A}(t) = \sum_{k=0}^{s-i} \mathbf{A}_k t^{-k} + \mathfrak{A}_s(t) t^{-s} \qquad (s = 0, 1, \ldots), \tag{2.2}$$

where the $\mathbf{A}_k$ are constant $n \times n$ matrices and the elements of $\mathfrak{A}_s(t)$ $(s = 1,2, \ldots)$ are holomorphic and uniformly bounded in some domain $\mathscr{D}_t$ extending to infinity. Then the system (2.1) can be transformed by a finite sequence of normalizing, zero-inducing, root-equalizing and shearing transformations to the canonical system

$$\frac{d\mathbf{W}}{dz} = z^r \mathfrak{C}(z)\mathbf{W}, \tag{2.3}$$

where $z = t^p$, r and $p > 0$ are integers,

$$z^r \mathfrak{C}(z) = \begin{bmatrix} \mathfrak{C}_1(z) & & & \\ & \mathfrak{C}_2(z) & & \\ & & \ddots & \\ & & & \mathfrak{C}_l(z) \end{bmatrix} + z^{-2} \begin{bmatrix} \mathfrak{B}_{11}(z)\mathfrak{B}_{12}(z) & \ldots & \mathfrak{B}_{1l}(z) \\ \mathfrak{B}_{21}(z)\mathfrak{B}_{22}(z) & \ldots & \mathfrak{B}_{2l}(z) \\ & \ddots & \\ \mathfrak{B}_{l1}(z)\mathfrak{B}_{l2}(z) & \ldots & \mathfrak{B}_{ll}(z) \end{bmatrix} \tag{2.4}$$

and

$$z^r \mathfrak{C}_i(z) = \sum_{k=0}^{r+1} \mathbf{C}_{ik} z^{r-k} \mathbf{I}_{\mu_i} + z^{-1} \mathbf{J}_{\mu_i}$$

$$i = 1, 2, \ldots, l; \qquad \mu_1 + \mu_2 + \ldots + \mu_l = n$$

$$\mathfrak{B}_{ij}(z) = \sum_{k=0}^{s-1} \mathbf{B}_{ijk} z^{-k} + \mathfrak{B}_{ijs}(z) z^{-s}, \tag{2.5}$$

The elements of the matrices $\mathfrak{B}_{ij}(z)$ are holomorphic for z in the domain $\mathscr{D}$ which corresponds to $\mathscr{D}_t$ for all z sufficiently large. Each element of $\mathfrak{B}_{ijs}(z)$ $(i, j = 1, 2, \ldots, l;\ s = 0, 1, \ldots)$ is uniformly bounded on any path $\mathscr{P}$ extending to ∞ in $\mathscr{D}$. If $r \leqq -2$ the matrices $\mathfrak{C}_i(z)$ $(i = 1, 2, \ldots l)$ are identically zero. If $r > -2$, the $\mathbf{I}_{\mu_i}$ are $\mu_i \times \mu_i$ unit matrices, and the $\mathbf{J}_{\mu_i}$ are $\mu_i \times \mu_i$ square matrices with zeros or ones or a mixture of zeros and ones in the first subdiagonal while all other elements of $\mathbf{J}_{\mu_i}$ are zero. No two of the matrices $\mathfrak{C}_i(z)$ $(i = 1, 2, \ldots, l)$ in (2.5) are identical. In particular, if $i \neq j$ and $c_{ik} = c_{jk}$ for $k = 0, 1, \ldots, r$, then corresponding to any particular fixed j $(j = 1, 2, \ldots, l)$ we have $-1 \leqq \mathrm{Re} c_{j,r+1} < 0$, $0 \leqq \mathrm{Re} c_{i,r+1} < 1$ and $c_{i,r+1} - c_{j,r+1} \neq 1$ $(i = 1, 2, \ldots, l;\ i \neq j)$.

The statement of the above theorem is different in two aspects from that given by Turrittin in [3]. In the first place Turrittin proves the theorem for the case when $\mathfrak{A}(t)$ has a convergent power series expansion about $t = 0$. Secondly, Turrittin's version of the last sentence of the theorem is: *In particular, if $i \neq j$ and $c_{ik} = c_{jk}$ for $k = 0, 1, \ldots, r$ then $c_{i,r+1} - c_{j,r+1}$ is not only not zero but is also not an integer.*

It is readily verified that all of the transformations: normalizing, zero-inducing, root-equalizing, and shearing preserve holomorphy of $\mathfrak{A}(t)$ for all t sufficiently large, and $t \in \mathscr{D}_t$ extending to infinity.

We now establish the statement in the last sentence of Theorem 2.1, assuming Turrittin's version of the theorem. Consider the subset of integers $\{i\}$ of the set $\{i=1, 2, \ldots, l\}$ such that with $i \in \{i\}$, and $i \neq j$, $c_{ik} - c_{jk} = 0$, $k = 0, 1, \ldots, r$. Corresponding to the points $c_{i,r+1}$ in the complex λ plane, we define integers k_i as follows: if $c_{i,r+1}$ lies in the strip $k_i \leqq \operatorname{Re} \lambda < k_i + 1$ we associate the integer k_i with $c_{i,r+1}$. Without loss of generality we assume that $\{i\} = \{1, 2, \ldots, \omega\}$ and that $k_1 \leqq k_2 \leqq \ldots \leqq k_\omega$. Let $\kappa = k_\omega - k_1$.

We apply the sequence of zero-inducing transformations

$$\mathbf{Z}_{k+1} = (\mathbf{I} + \mathbf{T}_k z^{-k}) \mathbf{Z}_k; \qquad k = 1, 2, \ldots, \kappa + 1; \qquad \mathbf{Z}_1 = \mathbf{W} \tag{2.6}$$

to (2.3), where $\mathbf{T}_k$ is an $n \times n$ matrix with an arbitrary leading $\omega \times \omega$ block and zeros everywhere else. Thus for example if

$$\frac{d\mathbf{Z}_{k-1}}{dz} = z^r [\mathbf{C}_0 + \mathbf{C}_1 z^{-1} + \ldots + \mathbf{C}_{r+1} z^{-r-1} + \mathbf{B}_0 z^{-r-2} + \ldots + \mathbf{B}_k z^{-k-r-1} + \ldots] \mathbf{Z}_{k-1} \tag{2.7}$$

where the leading $\omega \times \omega$ blocks of $\mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_{k-1}$ have already been filled with zeros, then

$$\frac{d\mathbf{Z}_k}{dz} = z^r [\mathbf{C}_0 + \mathbf{C}_1 z^{-1} + \ldots + \mathbf{C}_{r+1} z^{-r-1} + \mathbf{B}_0 z^{-r-2} + \ldots + \mathbf{B}_k^* z^{-k-r-1} + \ldots] \mathbf{Z}_k \tag{2.8}$$

where

$$\mathbf{B}_k^* = \mathbf{B}_k - \mathbf{T}_k \mathbf{C}_{r+1} + \mathbf{C}_{r+1} \mathbf{T}_k + k \mathbf{T}_k. \tag{2.9}$$

We divide the leading $\omega \times \omega$ blocks of the constant matrices $\mathbf{B}_k^*$, $\mathbf{B}_k$, $\mathbf{T}_k$ and $\mathbf{C}_{r+1}$ into smaller subblocks corresponding to the way in which $\mathbf{B}_0$ was subdivided in the above theorem. We denote these subblocks respectively by $\mathbf{B}_{ijk}^*$, $\mathbf{B}_{ijk}$, $\mathbf{T}_{ijk}$, and $\mathbf{C}_{ij,r+1}$. Equation (2.9) for $\mathbf{B}_{ijk}^*$ then becomes

$$\mathbf{B}_{ijk}^* = \mathbf{B}_{ijk} + (\mathbf{C}_i \mathbf{I}_{\mu_i} + \mathbf{J}_{\mu_i}) \mathbf{T}_{ijk} - \mathbf{T}_{ijk} (\mathbf{C}_j \mathbf{I}_{\mu_j} + \mathbf{J}_{\mu_j}) + k \mathbf{T}_{ijk}; \tag{2.10}$$

where $\mathbf{C}_i = \mathbf{C}_{i,r+1}$. Assuming Turrittin's version of Theorem 1, we have $\lambda_j - \lambda_i \neq k$. Elements of $\mathbf{T}_{ijk}$ can therefore be computed[4] starting from the right-hand vector, calculating from top to bottom and in this manner successively computing the vectors from right to left such that $\mathbf{B}_{ijk}^* = \mathbf{0}$. It is clear that each leading $\omega \times \omega$ block of $\mathbf{B}_k$ ($k = 0, 1, \ldots, \kappa$) can be filled with zeros in this manner.

The root-equalizing transformations

$$\mathbf{Z} = \begin{bmatrix} z^{k_1}\mathbf{I}_{\mu_1} & & & \\ & z^{k_2}\mathbf{I}_{\mu_2} & & \\ & & \ddots & \\ & & & z^{k}\mathbf{I}_{\mu_\omega} \end{bmatrix} \mathbf{Y}, \qquad \mathbf{Y} = \begin{bmatrix} \mathbf{I}_{\mu_1} & & & & \\ & \ddots & & & \\ & & \mathbf{I}_{\mu_{i-1}} & & \\ & & & z\mathbf{I}_{\mu_i} & \\ & & & & \ddots \\ & & & & & \mathbf{I}_{\mu_\omega} \end{bmatrix} \mathbf{W}$$

are now applied to (2.8) above, changing $\mathbf{C}_{i,r+1}$ ($i = 1, 2, \ldots, l$) to the form described in the last sentence of Theorem 2.1.

[4] For further details, see [3] p. 33.

The following theorem is also proved by Turrittin [3]. The main purpose of presenting a proof here is to obtain a recurrence relation for the coefficients of the solution matrix and to furnish an explicit algorithm for computing these coefficients.

THEOREM 2.2: *The system of differential equations in the canonical form* (2.3) *to* (2.5) *possesses a formal independent series solution matrix of the form*

$$\tilde{\mathbf{W}}(z) = \tilde{\mathbf{U}}(z)e^{\mathfrak{Q}(z)} \tag{2.11}$$

where

$$\tilde{\mathbf{U}}(z) = [\tilde{U}_{ij}(z)] \triangleq \sum_{k=0}^{\infty} \mathbf{U}_k z^{-k} \triangleq \sum_{k=0}^{\infty} [U_{ijk}] z^{-k}$$

$$\tilde{U}_{ij}(z) \text{ is } \mu_i \times \mu_j, \qquad \mathbf{U}_{ij0} = \delta_{ij}\mathbf{I}_{\mu_i}$$

$$\mathfrak{Q}(z) = [(\mathbf{I}_{\mu_i} q_i(z) + \mathbf{J}_{\mu_i} \log z)\delta_{ij}] = \sum_{k=0}^{r} \frac{\mathbf{Q}_k}{r+1-k} z^{r+1-k} + \mathbf{Q}_{r+1} \log z$$

$$q_i(z) = \sum_{k=0}^{r} \frac{q_{ik}}{r+1-k} z^{r+1-k} + q_{i,r+1} \log z \qquad (i, j = 1, 2, \ldots, l). \tag{2.12}$$

PROOF OF THEOREM 2.2: Let us put

$$z^r[\mathfrak{G}_i(z)\delta_{ij}] = \sum_{k=0}^{r+1} \mathbf{C}_k z^{r+1-k} \equiv z^r \mathbf{D}(z), \tag{2.13}$$

$$[\mathfrak{B}_{ij}(z)] = \mathfrak{B}(z) = \sum_{k=0}^{s-1} \mathbf{B}_k z^{-k} + \mathfrak{B}_s(z) z^{-s}. \tag{2.14}$$

We substitute (2.11) into (2.3), and use the fact that $\mathfrak{Q}$ and $\mathfrak{Q}'$ commute, to find

$$\tilde{\mathbf{U}}'(z) + \tilde{\mathbf{U}}(z)\mathfrak{Q}'(z) \triangleq [z^r \mathbf{D}(z) + z^{-2}\mathfrak{B}(z)]\tilde{\mathbf{U}}(z). \tag{2.15}$$

Expanding (2.3), using (2.12), (2.13), and (2.14) and equating equal powers of z, we obtain

$$\sum_{k=0}^{\min(\mu, r+1)} (\mathbf{C}_k \mathbf{U}_{\mu-k} - \mathbf{U}_{\mu-k}\mathbf{C}_k) + (\mu - r - 1)\mathbf{U}_{\mu-r-1} + \sum_{k=0}^{\mu-r-2} \mathbf{B}_k \mathbf{U}_{\mu-r-2-k} = \mathbf{O} \tag{2.16}$$

for $\mu = 0, 1, 2, \ldots$, where $\mathbf{U}_k = \mathbf{O}$ if $k < 0$ and $\mathbf{C}_k = \mathbf{Q}_k = \mathbf{O}$ if $k > r+1$.

The proof of Theorem 2.2 will be complete once we prove the following lemma.

LEMMA 2.1: *Under the assumption that* $\mathfrak{Q}(z)$ *has the form given by* (2.12) *with* $\mathbf{U}_0 = \mathbf{I}$, *equation* (2.16) *uniquely determines* $\mathbf{Q}_\mu (\mu = 0, 1, 2, \ldots, r+1)$ *and* $\mathbf{U}_\mu (\mu = 1, 2, 3, \ldots)$.

PROOF OF LEMMA 2.1: The proof of the lemma is by induction. With $\mu = 0$, (2.16) gives

$$\mathbf{C}_0 - \mathbf{Q}_0 = \mathbf{O}. \tag{2.17}$$

which has the unique solution $\mathbf{Q}_0 = \mathbf{C}_0$. With $\mu = 1$, (2.15) becomes

$$\mathbf{C}_1 - \mathbf{Q}_1 + \mathbf{C}_0\mathbf{U}_1 - \mathbf{U}_1\mathbf{C}_0 = \mathbf{O}. \tag{2.18}$$

Now if $\mathbf{U}_1$ is an arbitrary $n \times n$ matrix we know from the definition of $\mathbf{C}_0$ that the (i,i)th $(i=1,2, \ldots, l)$ blocks of $\mathbf{C}_0\mathbf{U}_1 - \mathbf{U}_1\mathbf{C}_0$ consisting of $\mu_i \times \mu_i$ matrices are identically zero. Hence these blocks of $\mathbf{C}_1 - \mathbf{Q}_1$ must also be zero, and therefore, $\mathbf{Q}_1 = \mathbf{C}_1$. Furthermore, it follows that all (i, j)th blocks of $\mathbf{U}_1$ for which $c_{i0} - c_{j0} \neq 0$ are uniquely determined and are zero. All (i, j)th blocks of $\mathbf{U}_1$ for which $c_{i0} = c_{j0}$ cannot be determined at this point. In particular, it is not possible to solve for $\mathbf{U}_{ii1} (i = 1.2. \ldots, l)$.

Assume that for some integer k, $0 \leqq k \leqq r$, we have completed solving (2.16) for all $\mu \leqq k$; that we have found $\mathbf{Q}_\mu = \mathbf{C}_\mu$ $(\mu = 0, 1, \ldots, k)$ and that we have determined all blocks of the matrices $\mathbf{U}_\mu$ as far as possible. More precisely, we make the following assumptions for each (i, j)th block of $\mathbf{U}_\mu$ $(i, j = 1, 2, \ldots, l;\ \mu = 0, 1, \ldots, k)$. If $c_{i\mu} - c_{j\mu} = 0$ $(\mu = 0, 1, \ldots, k-1)$ then assume $\mathbf{U}_{ij1}, \mathbf{U}_{ij2}, \ldots, \mathbf{U}_{ijk}$ are arbitrary, $\mu_i \times \mu_j$ matrices to be determined later. On the other hand, if p is the smallest integer in $0 \leqq p \leqq k$ for which $c_{ip} - c_{jp} \neq 0$, we assume that $\mathbf{U}_{ij0} = \mathbf{U}_{ij1} = \ldots = \mathbf{U}_{ij,\,k-p} = \mathbf{O}(\mu_i \times \mu_j)$, while $\mathbf{U}_{ij,\,k+1-p}, \ldots, \mathbf{U}_{ijk}$ are arbitrary $\mu_i \times \mu_j$ matrices.

For $\mu = k+1$, equation (2.16) becomes

$$\mathbf{C}_0\mathbf{U}_{k+1} - \mathbf{U}_{k+1}\mathbf{C}_0 + \ldots + \mathbf{C}_k\mathbf{U}_1 - \mathbf{U}_1\mathbf{C}_k + \mathbf{C}_{k+1} - \mathbf{Q}_{k+1} = \mathbf{O}. \tag{2.19}$$

If $\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_{k+1}$ are arbitrary $n \times n$ matrices, the (i, i)th $(i = 1, 2, \ldots, l)$ blocks of $\mathbf{C}_\mu\mathbf{U}_{k+1-\mu} - \mathbf{U}_{k+1-\mu}\mathbf{C}_\mu$ are zero when $\mu = 0, 1, \ldots, k$. Hence these blocks of $\mathbf{C}_{k+1} - \mathbf{Q}_{k+1}$ must also be zero, and therefore $\mathbf{Q}_{k+1} = \mathbf{C}_{k+1}$. On incorporating this result in (2.19) we find that the (i, j)th block of (2.19) becomes

$$(c_{i0} - c_{j0})\mathbf{U}_{ij,\,k+1} + (c_{i1} - c_{j1})\mathbf{U}_{ij,\,k} + \ldots + (c_{ik} - c_{jk})\mathbf{U}_{ij1} = \mathbf{O}. \tag{2.20}$$

If $c_{i\mu} - c_{j\mu} = 0$, $\mu = 0, 1, 2, \ldots, k$, then we cannot solve for $\mathbf{U}_{ij1}, \mathbf{U}_{ij2}, \ldots, \mathbf{U}_{ij,\,k+1}$. We leave these matrices undetermined for the moment. On the other hand, if p is the smallest nonnegative integer in $0 \leqq p \leqq k$ for which $c_{ip} - c_{jp} \neq 0$, then by the above induction hypothesis, (2.20) becomes $(c_{ip} - c_{jp})\mathbf{U}_{ij,\,k+1-p} = \mathbf{O}(\mu_i \times \mu_j)$, and so under the assumption that $\mathbf{U}_{ij,\,k+2-p}, \ldots, \mathbf{U}_{ij,\,k+1}$ are arbitrary $\mu_i \times \mu_j$ matrices we have $\mathbf{U}_{ij,\,k+1-p} = \mathbf{O}$.

It is clear at this point that $\mathbf{Q}_\mu$ is uniquely determined for each $\mu = 0, 1, \ldots, r+1$. We may therefore write the (i, j)th block of (2.16) in the form

$$\sum_{k=0}^{r+1} (q_{ik} - q_{jk})\mathbf{U}_{ij,\,\mu-k} + \mathbf{J}_{\mu_i}\mathbf{U}_{ij,\,\mu-r-1} - \mathbf{U}_{ij,\,\mu-r-1}\mathbf{J}_{\mu_j}$$

$$+ (\mu - r - 1)\mathbf{U}_{ij,\,\mu-r-1} + \sum_{k=0}^{\mu-r-2} \sum_{s=1}^{l} \mathbf{B}_{isk}\mathbf{U}_{sj,\,\mu-r-2-k} = \mathbf{O}. \tag{2.21}$$

For $\mu = r+2$, (2.21) becomes

$$\sum_{k=0}^{r+1} (q_{ik} - q_{jk})\mathbf{U}_{ij,\,r+2-k} + \mathbf{J}_{\mu_i}\mathbf{U}_{ij1} - \mathbf{U}_{ij1}\mathbf{J}_{\mu_j} + \mathbf{U}_{ij1} + \mathbf{B}_{ij0} = \mathbf{O}. \tag{2.22}$$

If $q_{i\mu} - q_{j\mu} = 0$ $(\mu = 0, 1, \ldots, r)$, and $\mathbf{U}_{ij,\,r+2}$ is an arbitrary $\mu_i \times \mu_j$ matrix, then

$$(1 + q_{i,\,r+1} - q_{j,\,r+1})\mathbf{U}_{ij1} + \mathbf{J}_{\mu_i}\mathbf{U}_{ij1} - \mathbf{U}_{ij1}\mathbf{J}_{\mu_j} + \mathbf{B}_{ij0} = \mathbf{O} \tag{2.23}$$

where, under the assumptions of Theorem 2.1, $q_{i,\,r+1} - q_{j,\,r+1} \neq 1$ if $i \neq j$. In either case, whether $i = j$, or $i \neq j$, eq (2.23) is of the same form as (2.10) and hence it uniquely determines $\mathbf{U}_{ij1}$. On the other hand, suppose that p is the smallest integer in $0 \leqq p \leqq r$ for which $q_{ip} - q_{jp} \neq 0$. Then

from the above analysis we know that $\mathbf{U}_{ij0}=\mathbf{U}_{ij1}=\cdots=\mathbf{U}_{ij,r+1-p}=\mathbf{O}$ $(\mu_i\times\mu_j)$. Thus if $\mathbf{U}_{ij,r+3-p}, \ldots, \mathbf{U}_{ij,r+2}$ are arbitrary $\mu_i\times\mu_j$ matrices, then

$$(q_{ip}-q_{jp})\mathbf{U}_{ij,r+2-p}+\mathbf{B}_{ijp}=\mathbf{O} \tag{2.24}$$

and this equation uniquely determines the hitherto unspecified $\mathbf{U}_{ij,r+2-p}$. Thus we see that the $\mu_i\times\mu_j$ matrices $\mathbf{U}_{ij1}$ are uniquely determined for $i, j=1, 2, \ldots, l$, and hence that $\mathbf{U}_1$ has been completely and uniquely determined.

Assume that for some integer $k\geqq 0$ we have completed solving (2.16) for $\mu=r+2, \ldots, r+2+k$; that we have uniquely determined $\mathbf{U}_1, \ldots, \mathbf{U}_{1+k}$ and that we have determined all blocks of the other matrices $\mathbf{U}_{1+s}$, $s>k$, as far as possible. More precisely, we make the following assumptions for each (i, j)th block $\mathbf{U}_{ij,1+\mu}$ of $\mathbf{U}_{1+\mu}$, $(0\leqq\mu\leqq r+2+k)$. If $q_{i\mu}-q_{j\mu}=0$, $\mu=0, 1, \ldots, r$, then assume that $\mathbf{U}_{ij1}, \ldots, \mathbf{U}_{ij,r+2+k}$ are arbitrary $\mu_i\times\mu_j$ matrices. On the other hand, if p is the smallest nonnegative integer in $0\leqq p\leqq r$ for which $q_{ip}-q_{jp}\neq 0$ then assume that $\mathbf{U}_{ij1}, \ldots, \mathbf{U}_{ij,r+2+k-p}$ have been uniquely determined while $\mathbf{U}_{ij,r+3+k-p}, \ldots, \mathbf{U}_{ij,r+2}$ are arbitrary $\mu_i\times\mu_j$ matrices, to be determined later.

With $\mu=r+3+k$, (2.21) becomes

$$\sum_{s=0}^{r+1}(q_{is}-q_{js})\mathbf{U}_{ij,r+3+k-s}+\mathbf{J}_{\mu_i}\mathbf{U}_{ij,2+k}-\mathbf{U}_{ij,2+k}\mathbf{J}_{\mu_j}+(k+2)\mathbf{U}_{ij,2+k}+\sum_{s=0}^{k+1}\sum_{\nu=1}^{l}\mathbf{B}_{i\nu s}\mathbf{U}_{\nu j,k+1-s}=\mathbf{O}. \tag{2.25}$$

If $q_{is}-q_{js}=0$ $(s=0, 1, \ldots, r)$, then with the assumption that $\mathbf{U}_{ij,r+3+k}$ is an arbitrary $\mu_i\times\mu_j$ matrix, the above induction hypothesis, and the assumptions of Theorem 2.1, we see that (2.25) is an equation in $\mathbf{U}_{ij,k+2}$ of exactly the same type as (2.10) was in $\mathbf{T}_{ijk}$, and thus uniquely determines $\mathbf{U}_{ij,k+2}$. On the other hand, suppose that p is the smallest nonnegative integer in $0\leqq p\leqq r$ for which $q_{ip}-q_{jp}\neq 0$. Then assuming $\mathbf{U}_{ij,r+4+k-p}, \ldots, \mathbf{U}_{ij,r+3+k}$ to be arbitrary $\mu_i\times\mu_j$ matrices, we see that eq (2.25) may be written

$$(q_{ip}-q_{jp})\mathbf{U}_{ij,r+3+k-p}=-\sum_{s=p+1}^{r+1}(q_{is}-q_{js})\mathbf{U}_{ij,r+3+k-s}-(k+2)\mathbf{U}_{ij,k+2}-\sum_{s=0}^{k+1}\sum_{\nu=1}^{l}\mathbf{B}_{i\nu s}\mathbf{U}_{\nu j,k+1-s}. \tag{2.26}$$

By the induction hypothesis, all matrices on the right of (2.26) are known; hence $\mathbf{U}_{ij,r+3+k-p}$ is uniquely determined.

Thus the matrices $\mathbf{U}_\mu(\mu=1,2, \ldots)$ are uniquely determined by eq (2.16).

The above proof can be easily modified for purposes of computing the jth column of blocks only. Suppose we are interested in knowing the coefficients of a particular vector solution, say the νth column $\widetilde{\mathbf{W}}_j^\nu(z)$ in the jth column of blocks $\widetilde{\mathbf{W}}_j(z)$ of $\widetilde{\mathbf{W}}(z)$. If $\mathbf{J}_{\mu_j}$ has a one in its νth column as well as in its $(\nu+1)$th, $\ldots$, $(\mu-1)$th but not in its μth column where $\mu\leqq\mu_j$, then it is readily shown that

$$\widetilde{\mathbf{W}}_j^\nu(z)\hat{=}\sum_{k=0}^{\infty}\left\{\sum_{s=\nu}^{\mu}\mathbf{U}_{jk}^s\frac{(\log z)^{s-\nu}}{(s-\nu)!}\right\}z^{-k}e^{q_j(z)}, \tag{2.27}$$

where $\mathbf{U}_{jk}^s$ is the sth $(s=1,2, \ldots, \mu_j)$ column vector in the jth column of blocks of the matrix $\mathbf{U}_k$. Thus to obtain the νth formal solution vector in the jth column of blocks it is only necessary to compute the μth, $(\mu-1)$th, $\ldots$, νth columns of $\mathbf{U}_{jk}$ at each stage.

3. Error Bounds for Formal Partial Sum Approximations to Actual Solutions

We shall obtain an actual solution vector of the system (2.3) to (2.5) corresponding to νth formal solution vector in the jth column of blocks of formal solutions (eq (2.27)), and we shall obtain error bounds for the difference between a partial sum of (2.27) and the corresponding actual solution vector.

3.1. The differential Equation for an Approximation

Starting with the partial sum

$$\Phi_{jm}(z)=\sum_{k=0}^{m-1}\mathbf{U}_{jk}z^{-k}e^{\mathfrak{Q}_j(z)} \tag{3.1}$$

where in the notation of Theorem 2.2, $\mathbf{U}_{jk}$ is the jth column of blocks of the matrix $\mathbf{U}_k$ and

$$\mathfrak{Q}_j(z)=q_j(z)\mathbf{I}_{\mu_j}+\mathbf{J}_{\mu_j}\log z, \tag{3.2}$$

we define $\mathbf{R}_{jm}(z)$ by

$$\frac{d}{dz}\Phi_{jm}(z)-[\mathfrak{Q}'(z)+z^{-2}\mathfrak{B}(z)]\Phi_{jm}(z)=\mathbf{R}_{jm}(z)e^{\mathfrak{Q}_j(z)} \tag{3.3}$$

On expanding the (i, j)th block of (3.3) using (2.14) and (2.25) and denoting $q_i(z)-q_j(z)=q_{ij}(z)$, we obtain the following expression for $\mathbf{R}_{ijm}(z)$

$$\begin{aligned}&q'_{ij}(z)\mathbf{U}_{ijm}z^{-m}-[q'_{ij}(z)-(q_{ip}-q_{jp})z^{r-p}]\mathbf{U}_{ijm}z^{-m}\\&+\sum_{\mu=m+p+1}^{m+r+1}\Bigg[\sum_{s=\mu+1-m}^{\min(\mu,\,r+1)}(q_{is}-q_{js})\mathbf{U}_{ij,\,\mu-s}+\mathbf{J}_{\mu_i}\mathbf{U}_{ij,\,\mu-r-1}-\mathbf{U}_{ij,\,\mu-r-1}\mathbf{J}_{\mu_j}\\&+(\mu-r-1)\mathbf{U}_{ij,\,\mu-r-1}+\sum_{s=0}^{\mu-r-2}\{\mathbf{B}_s\mathbf{U}_{j,\,\mu-r-2-s}\}_i\Bigg]z^{r-\mu}-\sum_{s=0}^{m-1}\{\mathfrak{B}_{m-s}(z)\mathbf{U}_{js}\}_i z^{-m-s}\end{aligned} \tag{3.4}$$

if $i\neq j$, and for some p in $0\leqq p\leqq r$, $q_{ip}-q_{jp}\not\equiv 0$. On the other hand, if $q_{ip}=q_{jp}$, $0\leqq p\leqq r$, then

$$\mathbf{R}_{ijm}(z)=[(q_{i,\,r+1}-q_{j,\,r+1}+m)\mathbf{U}_{ijm}+\mathbf{J}_{\mu_i}\mathbf{U}_{ijm}-\mathbf{U}_{ijm}\mathbf{J}_{\mu_j}]z^{-m-1}-\sum_{s=0}^{m-1}\{\mathfrak{B}_{m-s}(z)\mathbf{U}_{js}\}_i z^{-m-s}, \tag{3.5}$$

where (3.5) includes the case $i=j$.

3.2. The Various Possible Cases

It is demonstrated in [4] that for each fixed j ($j=1, 2, \ldots, l$) the complete vicinity of infinity can be divided into a finite number N, say, of sectors $\mathscr{S}_{kj}$ ($k=1, 2, \ldots, N$) such that for all i ($i=1, 2, \ldots, l$) for which $q_{ip}-q_{jp}\not\equiv 0$ for some integer p in $0\leqq p\leqq r$, we have either Re $q_{ij}(z)$ bounded above, or else Re $q_{ij}(z)$ tending to $+\infty$ as $z\to\infty$ in a particular sector $\mathscr{S}_{kj}$. This fact enables us to classify the various ways of constructing actual solution vectors corresponding to formal solution vectors under two different headings: (a) those cases for which it is possible to define an actual solution vector by a single Volterra integral equation; and (b) those cases for which we require a simultaneous pair of Volterra integral equations to define a solution vector.

Let us establish our notation for purposes of characterizing cases (a) and (b) above. The fixed limits of integration in the closure, $\overline{\mathscr{D}}$, of the domain $\mathscr{D}$ in which $\mathfrak{Q}(z)$ is holomorphic will

be denoted by $\{\zeta\}$. In case (a) $\{\zeta\}$ consists of a single point ζ, say, and in case (b) $\{\zeta\}$ consists of two points, ζ_1, and ζ_2, say. $\mathscr{E}_a$ will denote a closed disc with centre at the origin and radius a. If for all integers i in the range $1 \leqq i \leqq l$ $q_{ip} = q_{jp}$ for $p = 0, 1, \ldots, r$, then $a = 0$ provided that every member of $\{\zeta\}$ is unbounded. , Put $\chi = \min_{\zeta \in \{\zeta\}} |\zeta|$. If $\chi < \infty$ we set $a = \chi$. If, in case (b), all members of $\{\zeta\}$ are bounded then we assume that $\zeta_1 \neq \zeta_2$ and $|\zeta_1| = |\zeta_2|$. If, for at least one i in the range $1 \leqq i \leqq l$, we have $q_{ip} \neq q_{jp}$ for some $p \in (0, 1, \ldots, r)$, we let a_0 denote the maximum modulus of the set of zeros of $q'_{ij}(z)$ for all such i, and set $a = a_0$ if $\chi = \infty$. If $\chi < \infty$ chose $\{\zeta\}$ in such a way that $\chi \geqq a_0$ and no member of $\{\zeta\}$ coincides with a zero of $q'_{ij}(z)$. Let $\mathscr{D}^* = \mathscr{D} - \mathscr{E}_a - \mathscr{S}$ where $\mathscr{S}$ is empty or $\{\infty\}$ according to whether or not $\mathfrak{Q}(z) = \mathbf{O}$. The paths $\mathscr{P}$ are defined as in [1].

Case (a) above is related to the extreme eigenvalue case in [1]. *It is characterized as follows.* We define a region $\mathscr{D}(z, \zeta)$ to be the union of all points z, other than ζ, such that there is a path $\mathscr{P}$ connecting $z \in \mathscr{D}(z, \zeta)$ and ζ satisfying the following *extreme eigenvalue conditions:*

(1) If $\zeta \in \overline{\mathscr{D}^*} - \mathscr{D}^*$, then $\mathscr{P}$ lies entirely in $\mathscr{D}^*$ except for ζ.

(2) For given ζ and z, all elements of the matrices $\exp\{\mathfrak{Q}_i(\tau) - \mathfrak{Q}_i(t) - \mathbf{I}_{\mu_i}[q_j(\tau) - q_j(t)]\}$ $(i = 1, 2, \ldots, l;\ i \neq j)$ are bounded for all points t, τ in the order ζ, t, τ, z on $\mathscr{P}$. If $\mathbf{J}_{\mu_j} \neq \mathbf{O}$, or if for some i $(i = 1, 2, \ldots, l;\ i \neq j)$ we have $q_{ip} = q_{jp}$ for $p = 0, 1, \ldots, r$, we require that τ/t be bounded.

(3) $V_{\mathscr{P}}(t^{\eta-1})$ is bounded, where the fixed number η may be taken to be zero if $\mathbf{J}_{\mu_j} = \mathbf{O}$ but otherwise η is an arbitrary positive number less than 1.

Case (b) above is related to the interior eigenvalue case in [1]. *It is characterized as follows.* We define a region $\mathscr{D}(\zeta_1, \zeta_2, \eta)$ to be the union of the points z for which there is a path $\mathscr{P}$ connecting ζ_1, z and ζ_2 (in that order) satisfying the following *interior eigenvalue conditions:*

(1) Except possibly for ζ_1 and ζ_2, $\mathscr{P}$ lies entirely in $\mathscr{D}^*$.

(2) The integers $i = 1, 2, \ldots, l$ are subdivided into two nonempty disjoint classes $\mathbf{N}_1$ and $\mathbf{N}_2$ such that if ζ_1, and ζ_2 are given, then for all points t and τ in the order ζ_1, t, τ, ζ_2 on $\mathscr{P}$, the elements of the matrix $\mathfrak{Q}_{ij}(\tau, t) = \exp\{\mathfrak{Q}_i(\tau) - \mathfrak{Q}_i(t) - \mathbf{I}_{\mu_i}[q_j(\tau) - q_j(t)]\}$ are bounded if $i \in \mathbf{N}_1$ $(i \neq j)$ while if $i \in \mathbf{N}_2$ $(i \neq j)$ then all elements of the matrix $\mathfrak{Q}_{ij}(t, \tau)$ are bounded. If $\mathbf{J}_{\mu_j} \neq \mathbf{O}$, or if for some i in the range $(1, l)$ $(i \neq j)$ we have $q_{ip} = q_{jp}$ for $p = 0, 1, \ldots, r$, then we assume without loss of generality that both i and j are in $\mathbf{N}_2$, that ζ_1 is bounded, and that t/τ remains bounded for all points t, τ in the order ζ_1, t, τ, ζ_2 on $\mathscr{P}$.

(3) $\frac{1}{2}\{a_1 + a_4 + \sqrt{(a_1 - a_4)^2 + 4a_2a_3}\}V_{\mathscr{P}}(t^{\eta-1}) < 1$, where η may be taken zero if $\mathbf{J}_{\mu_j} = \mathbf{O}$, but otherwise η is an arbitrary positive number less than 1, and a_1, a_2, a_3 and a_4 are defined by (3.31).

We observe that it is no longer always possible to choose the points $\{\zeta\}$ at infinity, in contrast to the cases in [1].

If $r \geqq 0$ we can again represent the eigenvalues of $\mathbf{C}_0$ as points in the complex plane, and enclose them by the smallest strictly convex closed polygon. As opposed to the distinct eigenvalue case it is now possible to have several formal solution vectors corresponding to a particular eigenvalue q_{0j} of $\mathbf{C}_0$. Nevertheless, if q_{0j} is an interior point of the polygon and if at least one endpoint of integration is taken at infinity [5] it will be necessary to express the error vector by a simultaneous pair of Volterra vector integral equations. If q_{0j} is an extreme point of the polygon and only one formal vector solution corresponds to q_{0j}, it is again possible, as in the distinct eigenvalue case, to express the error vector by a single Volterra vector integral equation. If q_{0j} is an extreme point of the polygon and more than one formal vector solution corresponds to q_{0j}, or if q_{0j} is any other boundary point of the polygon, we can always determine whether we can express the error vector by a single or a simultaneous pair of Volterra integral equations by testing whether or not the above extreme eigenvalue conditions can be satisfied.

[5] We are here attempting to construct actual solutions for which the formal solutions are asymptotic expansions.

We are now prepared to set up the integral equations. Let a vector $\mathbf{W}_j^\nu(z)$ of holomorphic functions satisfy the differential equation (2.3). Denoting the νth column vector of Φ_{jm} by Φ_{jm}^ν, we find from (3.3) and (2.27) that the error vector

$$\boldsymbol{\epsilon}_{jm}^\nu(z)=(\mathbf{W}_j^\nu(z)-\Phi_{jm}^\nu(z))e^{-q_j(z)} \tag{3.6}$$

satisfies the differential equation

$$\frac{d}{dz}\boldsymbol{\epsilon}_{jm}^\nu(z)-[\mathbf{D}_j'(z)+z^{-2}\mathfrak{B}(z)]\boldsymbol{\epsilon}_{jm}^\nu(z)=-\sum_{s=\nu}^{\mu}\mathbf{R}_{jm}^s(z)\frac{(\log z)^{s-\nu}}{(s-\nu)!} \tag{3.7}$$

where $\mathbf{R}_{jm}^s(z)$ is the sth column vector of $\mathbf{R}_{jm}(z)$, and $\mathbf{D}_j(z)=\mathfrak{Q}(z)-\mathbf{I}q_j(z)$.

Hence for the extreme eigenvalue case, if the vector $\boldsymbol{\epsilon}_{jm}^\nu(z)$ satisfies

$$\boldsymbol{\epsilon}_{jm}^\nu(z)=\int_\zeta^z e^{\mathbf{D}_j(z)-\mathbf{D}_j(t)}t^{-2}\mathfrak{B}(t)\boldsymbol{\epsilon}_{jm}^\nu(t)dt+\mathbf{R}_{jm}^{\nu*}(z) \tag{3.8}$$

where $\mathbf{R}_{jm}^{\nu*}(z)$ is defined by

$$\mathbf{R}_{jm}^{\nu*}(z)=-\int_\zeta^z e^{\mathbf{D}_j(z)-\mathbf{D}_j(t)}\sum_{s=\nu}^{\mu}\mathbf{R}_{jm}^s(t)\frac{(\log t)^{s-\nu}}{(s-\nu)!}\,dt, \tag{3.9}$$

and the path $\mathscr{P}$ of integration satisfies the above extreme eigenvalue conditions, then $\boldsymbol{\epsilon}_{jm}^\nu(z)$ simultaneously satisfies (3.7).

For the interior eigenvalue case we partition $\boldsymbol{\epsilon}_{jm}^\nu(z)$ into $\boldsymbol{\epsilon}_{jm1}^\nu(z)$ and $\boldsymbol{\epsilon}_{jm2}^\nu(z)$, $\mathbf{D}_j(z)$ into $\mathbf{D}_{j1}(z)$ and $\mathbf{D}_{j2}(z)$ and $\mathbf{R}_{jm}^s$ into $\mathbf{R}_{jm1}^s(z)$, $\mathbf{R}_{jm2}^s(z)$, and in a corresponding manner $\mathfrak{B}(z)$ into $\mathfrak{a}_1(z)$, $\mathfrak{a}_2(z)$, $\mathfrak{a}_3(z)$ and $\mathfrak{a}_4(z)$ so that (3.7) may be written

$$\frac{d}{dz}\begin{pmatrix}\boldsymbol{\epsilon}_{jm1}^\nu(z)\\ \boldsymbol{\epsilon}_{jm2}^\nu(z)\end{pmatrix}-\left[\begin{pmatrix}\mathbf{D}_{j1}'(z)&\mathbf{O}\\ \mathbf{O}&\mathbf{D}_{j2}'(z)\end{pmatrix}+z^{-2}\begin{pmatrix}\mathfrak{a}_1(z)&\mathfrak{a}_2(z)\\ \mathfrak{a}_3(z)&\mathfrak{a}_4(z)\end{pmatrix}\right]\begin{pmatrix}\boldsymbol{\epsilon}_{jm1}^\nu(z)\\ \boldsymbol{\epsilon}_{jm2}^\nu(z)\end{pmatrix}=-\sum_{s=\nu}^{\mu}\begin{pmatrix}\mathbf{R}_{jm1}^s(z)\\ \mathbf{R}_{jm2}^s(z)\end{pmatrix}\frac{(\log z)^{s-\nu}}{(s-\nu)!}. \tag{3.10}$$

In (3.10) $\boldsymbol{\epsilon}_{jm1}^\nu(z)$ contains all the ith elements of $\boldsymbol{\epsilon}_{jm}^\nu(z)$ for which $i\in\mathbf{N}_1$, and $\boldsymbol{\epsilon}_{jm2}^\nu(z)$ contains all the ith elements for which $i\in\mathbf{N}_2$. It follows that if $\boldsymbol{\epsilon}_{jm}^\nu(z)$ satisfies

$$\begin{pmatrix}\boldsymbol{\epsilon}_{jm1}^\nu(z)\\ \boldsymbol{\epsilon}_{jm2}^\nu(z)\end{pmatrix}=\begin{pmatrix}\int_{\zeta_1}^z&\mathbf{0}\\ \mathbf{0}&\int_{\zeta_2}^z\end{pmatrix}\left[\begin{pmatrix}e^{\mathbf{D}_{j1}(z)-\mathbf{D}_{j1}(t)}&\mathbf{O}\\ \mathbf{O}&e^{\mathbf{D}_{j2}(z)-\mathbf{D}_{j2}(t)}\end{pmatrix}t^{-2}\begin{pmatrix}\mathfrak{a}_1(t)&\mathfrak{a}_2(t)\\ \mathfrak{a}_3(t)&\mathfrak{a}_4(t)\end{pmatrix}\begin{pmatrix}\boldsymbol{\epsilon}_{jm1}^\nu(t)\\ \boldsymbol{\epsilon}_{jm2}^\nu(t)\end{pmatrix}\right]dt+\begin{pmatrix}\mathbf{R}_{jm1}^{\nu*}(z)\\ \mathbf{R}_{jm2}^{\nu*}(z)\end{pmatrix} \tag{3.11}$$

where

$$\mathbf{R}_{jmk}^{\nu*}(z)=-\int_{\zeta_k}^z e^{\mathbf{D}_{jk}(z)-\mathbf{D}_{jk}(t)}\sum_{s=\nu}^{\mu}\mathbf{R}_{jmk}^s(t)\frac{(\log t)^{s-\nu}}{(s-\nu)!}\,dt,\ k=1,2, \tag{3.12}$$

and the path of integration satisfies the above interior eigenvalue conditions, then $\boldsymbol{\epsilon}_{jm}^\nu(z)$ also satisfies (3.7).

3.3. Treatment of $\mathbf{R}_{ijm}(z)$

We shall derive expressions for the elements of the matrices (3.9) and (3.12) which will enable us to obtain good bounds in the following sections. Let us consider the block [6]

$$\mathbf{R}^*_{ijm}(z) = -\int_\zeta^z e^{\Omega_i(z)-\Omega_i(t)}\mathbf{R}_{ijm}(t)e^{\Omega_j(t)-q_j(z)}dt. \tag{3.13}$$

We split (3.13) into its dominant part plus a remainder; thus we express $\mathbf{R}_{ijm}(t)$ in the form

$$\mathbf{R}_{ijm}(t) = \left[\Omega_i'(t)\mathbf{U}_{ijm} + \frac{m}{t}\mathbf{U}_{ijm} - \mathbf{U}_{ijm}\Omega_j'(t)\right]t^{-m} + \mathbf{P}_{ijm}(t) - \sum_{k=0}^{m-1}\{\mathfrak{B}_{m-k}(t)\mathbf{U}_{jk}\}_i t^{-m-2} \tag{3.14}$$

where

$$\begin{aligned}\mathbf{P}_{ijm}(t) = -[&[(q_{i,p+1}-q_{j,p+1})t^{r-p-1}+\ldots+(q_{i,r+1}-q_{j,r+1})t^{-1}+mt^{-1}+\mathbf{J}_{\mu_i}t^{-1}]\mathbf{U}_{ijm}-U_{ijm}\mathbf{J}_{\mu_j}t^{-1}]t^{-m}\\ &-\sum_{\mu=m+p+1}^{m+r+1}\left[\sum_{k=\mu+1-m}^{\min\,(r+1,\,\mu)}(q_{ik}-q_{jk})\mathbf{U}_{ij,\,\mu-k}+\mathbf{J}_{\mu_i}U_{ij,\,\mu-r-1}-U_{ij,\,\mu-r-1}\mathbf{J}_{\mu_j}\right.\\ &\left.+(\mu-r-1)U_{ij,\,\mu-r-1}+\sum_{k=0}^{\mu-r-2}\{B_kU_{j,\,\mu-r-2-k}\}_i\right]t^{r-\mu}.\end{aligned} \tag{3.15}$$

The form of the first term in (3.14) is such that when substituted into (3.13) it can be readily integrated by parts. We define $\mathbf{S}_{ij}(z,\,t)$ by

$$\mathbf{S}_{ij}(z,\,t)e^{q_j(z)-q_j(t)} = e^{\Omega_i(z)-\Omega_i(t)}\,\mathbf{P}_{ijm}(t)e^{\Omega_j(t)-q_j(z)} \tag{3.16}$$

Clearly, knowing the form of $\Omega_i(t)$, we can express the matrix $\mathbf{S}_{ij}(z,\,t)$ explicitly. In any case $\mathbf{S}_{ij}(z,\,t)$ is simply $\mathbf{P}_{ijm}(t)$ multiplied by a matrix polynomial in $\log(z/t)$. On substituting (3.14) into (3.13), integrating by parts and making use of (3.16), we find that

$$\mathbf{R}^*_{ijm}(z) = \mathbf{U}^*_{ijm}(z)\,(z^{\eta_1-m}-\zeta^{\eta_1-m}) + \mathbf{V}^1_{ijm}(z)\,(z^{\eta_1-m-1}-\zeta^{\eta_1-m-1}) - \int_\zeta^z \mathbf{V}^2_{ijm}(z,\,t)\,(m+1-\eta_1)t^{-(m+2-\eta_1)}dt, \tag{3.17}$$

where η_1 may be taken zero if $\mathbf{J}_{\mu_i}=\mathbf{O}$ for all $i=1,\,2,\,\ldots,\,l$, otherwise it is an arbitrary positive number less than 1. The matrices $\mathbf{U}^*_{ijm}(z)$, $\mathbf{V}^1_{ijm}(z)$ and $\mathbf{V}^2_{ijm}(z,\,t)$ are defined by

$$\mathbf{U}^*_{ijm}(z) = \frac{\mathbf{U}_{ijm}z^{-m}e^{\Omega_j(z)-q_j(z)} - e^{\Omega_i(z)-\Omega_i(\zeta)}\mathbf{U}_{ijm}\zeta^{-m}e^{\Omega_j(\zeta)-q_j(z)}}{z^{\eta_1-m}-\zeta^{\eta_1-m}} \tag{3.18}$$

$$\mathbf{V}^1_{ijm}(z) = \frac{\mathbf{S}_{ij}(z,\,z)/q_{ij}'(z) - e^{q_{ij}(z)-q_{ij}(\zeta)}\mathbf{S}_{ij}(z,\,\zeta)/q_{ij}'(\zeta)}{z^{\eta_1-m-1}-\zeta^{\eta_1-m-1}}, \tag{3.19}$$

[6] Here and henceforth we interpret $e^{\Omega_j(t)-q_j(t)} = e^{\Omega_j(t)-\mathbf{I}_{n_j}q_j(t)}$.

and

$$\mathbf{V}^2_{ijm}(z,\,t)=\frac{t^{-\eta_1}}{m+1-\eta_1}\left[\frac{\partial}{\partial t}\left(\frac{\mathbf{S}_{ij}(z,\,t)}{q'_{ij}(t)}\right)e^{q_{ij}(z)-q_{ij}(t)}t^{m+2}-e^{\mathbb{Q}_{ik}(z)-\mathbb{Q}_{ik}(t)}\sum_{k=1}^{m-1}\{\mathfrak{B}_{m-k}(t)\mathbf{U}_{jk}\}_i e^{\mathbb{Q}_{jk}(t)-q_{ij}(z)}\right], \tag{3.20}$$

Equations (3.19) and (3.20) hold, provided that there is an integer p in $0\leqq p\leqq r$ such that $q_{ip}-q_{jp}\neq 0$. If this is not so we define $\mathbf{V}^1_{ijm}(z)=\mathrm{O}$, and

$$\mathbf{V}^2_{ijm}(z,\,t)=\frac{t^{-\eta_1}}{m+1-\eta_1}\left[e^{\mathbb{Q}_{ik}(z)-\mathbb{Q}_{ik}(t)}\sum_{k=0}^{m-1}\{\mathfrak{B}_{m-k}(t)\mathbf{U}_{jk}\}_i e^{\mathbb{Q}_{ik}(t)-q_{ij}(z)}\right]. \tag{3.21}$$

The representation (3.17) was motivated by the following consideration. We observe that each element of $\mathbf{R}^*_{ijm}(z)$ is of the form

$$p(z,\,\zeta)=f(z)-f(\zeta)+g(z)-g(\zeta)+\int_z^z h(z,\,t)t^{-m-2}dt, \tag{3.22}$$

where $|f(z)|=\mathrm{O}(|z|^{-m}|\log z|^{\mu-\nu})$ (see eq (2.27)), . $|g(z)|=\mathrm{O}(|z|^{-m-1}|\log z|^{\mu-\nu})$ and $h(z,\,t)=\mathrm{O}(|\log z|^k|\log t|^s)$, as $|z|,\,|t|\to\infty$, where k and s are integers. The reason for introducing terms like $t^{-\eta}$, $t^{-\eta_1}$ where η and η_1 are nonnegative numbers less than 1 is to eliminate logarithmic terms conveniently, and at the same time obtain good bounds. From (3.22), we have

$$\begin{aligned}
|p(z,\,\zeta)|\leqq&\left|\frac{f(z)-f(\zeta)}{z^{\eta_1-m}-\zeta^{\eta_1-m}}\right||z^{\eta_1-m}-\zeta^{\eta_1-m}|\\
&+\left|\frac{g(z)-g(\zeta)}{z^{\eta_1-m-1}-\zeta^{\eta_1-m-1}}(z^{\eta_1-m-1}-\zeta^{\eta_1-m-1})+\int_\zeta^z h(z,\,t)t^{-\eta_1}t^{\eta_1-m-2}dt\right|\\
=&\left|\frac{f(z)-f(\zeta)}{z^{\eta_1-m}-\zeta^{\eta_1-m}}\right|\left|(\eta_1-m)\int_\zeta^z t^{\eta_1-m-1}dt\right|\\
&+\left|\int_\zeta^z\left\{\frac{(\eta_1-m-1)(g(z)-g(\zeta))}{z^{\eta_1-m-1}-\zeta^{\eta_1-m-1}}+h(z,\,t)t^{-\eta_1}\right\}t^{\eta_1-m-2}dt\right|,
\end{aligned}$$

from which

$$|p(z,\,\zeta)|\leqq\sup_{t\in\mathscr{P}}\left|\frac{f(t)-f(\zeta)}{t^{\eta_1-m}-\zeta^{\eta_1-m}}\right|\mathscr{V}_{\mathscr{P}}\,(t^{\eta_1-m})+\sup_{t,\,\tau\in\mathscr{P}}\left|\frac{g(\tau)-g(\zeta)}{\tau^{\eta_1-m-1}-\zeta^{\eta_1-m-1}}+\frac{h(\tau,\,t)t^{-\eta_1}}{\tau^{\eta_1-m-1}}\right|\mathscr{V}_{\mathscr{P}}\,(t^{\eta_1-m-1}) \tag{3.23}$$

where in the last supremum t and τ are points on $\mathscr{P}$ in order $\zeta,\,t,\,\tau,\,z$.

In each of the next two sections we first establish the existence and uniqueness of the solutions to the integral equations (3.8) and (3.11), then we obtain the error bounds. Our procedure is, in one aspect, similar to that in [6] in that we first establish that the conditions in sections 3.2 determine a domain. We then appeal to the contraction mapping principle (see e.g. [8], sec. 11) to establish existence and uniqueness. Proofs by use of successive approximations are given in [7].

3.4. Error Bounds for the Extreme Eigenvalue Case

We use the following notation to refer to the elements of a partitioned $n\times n$ matrix $\mathbf{A}$. $\mathbf{A}_j=\{\mathbf{A}\}_j$ will denote the jth column of blocks in $\mathbf{A}$, $\mathbf{A}_{ij}=\{\mathbf{A}_j\}_i=\{\mathbf{A}\}_{ij}$ the $(i,\,j)$th block, $\mathbf{A}^\nu_i=\{\mathbf{A}_j\}^\nu$

the νth vector in the jth column of blocks, and $A_{ij}^{\omega\nu}=\{A_j\}_i^{\omega\nu}=\{A_j^\nu\}_i^\omega$ the (ω,ν)th element in the (i,j)th block.

LEMMA 3.1: *If under the extreme eigenvalue conditions of section* 3.2, $\mathscr{D}(z,\zeta)$ *contains at least one finite point* z^*, *distinct from* ζ, *then* $\mathscr{D}(z,\zeta)$ *is a domain, that is, an open connected set of points.*

PROOF: The region $\mathscr{D}^*$ defined in Section 3.2 is certainly a domain. By assumption there is a path $\mathscr{P}$ in $\mathscr{D}^*$ connecting z^* with ζ satisfying the three extreme eigenvalue conditions. Similarly there is such a path connecting an arbitrary finite point z_0 on $\mathscr{P}$ with ζ. We can clearly choose a sufficiently small positive number ρ such that the disk $\mathscr{S}(\rho)$: $|z-z_0|<\rho$ lies entirely in $\mathscr{D}^*$, and when an arbitrary point z_1 of the disk is joined to z_0 by a straight line $\mathscr{P}_1$ then conditions (1) and (3) are satisfied on $\mathscr{P}\cup\mathscr{P}_1$. It remains to be shown that condition (2) is also satisfied.

To show this we define matrices

$$\mathbf{K}_i(\mathscr{P})=\sup_{\tau,\,t\in\mathscr{P}}|\exp\,[\mathfrak{Q}_i(\tau)-\mathfrak{Q}_i(t)-\mathbf{I}_{\mu_i}(q_j(\tau)-q_j(t))]|\qquad(i=1,\,2,\,\ldots,\,l\,;\,i\neq j)\tag{3.24}$$

$$K_i'(\mathscr{P}_1)=\sup_{\tau_1,\,t_1\in\mathscr{P}_1}|\exp[\mathfrak{Q}_i(\tau_1)-\mathfrak{Q}_i(t_1)-\mathbf{I}_{\mu_i}q_j(\tau_1)-q_j(t_1))]|,\tag{3.25}$$

where τ, t, τ_1, t_1 are points on $\mathscr{P}\cup\mathscr{P}_1$ in the order z_1, τ_1, t_1, z_0, τ, t, ζ, and the suprema apply to individual elements of the matrices, as in [1]. Clearly, by hypothesis each of these matrices has nonnegative bounded elements. We now choose τ and t on $\mathscr{P}\cup\mathscr{P}_1$ in the order z_1, τ, t, ζ. We then consider

$$K_i(\mathscr{P}\cup\mathscr{P}_1)=\sup_{\tau,\,t\in\mathscr{P}\cup\mathscr{P}_1}|\exp\,[\mathfrak{Q}_i(\tau)-\mathfrak{Q}_i(t)-\mathbf{I}_{\mu_i}(q_j(\tau)-q_j(t))]|$$

where i is a part of the range $(1,\,l)$ other than j. If τ is on $\mathscr{P}$ then so is t, and in this case by hypothesis the matrix on the right consists of nonnegative bounded elements. Similarly when τ and t are on $\mathscr{P}_1$. Suppose then, that τ is on $\mathscr{P}_1$ and t is on $\mathscr{P}$. In this case

$$|\exp\,[\mathfrak{Q}_i(\tau)-\mathfrak{Q}_i(t)-\mathbf{I}_{\mu_i}(q_j(\tau)-q_j(t))]|=|\exp\,[\mathfrak{Q}_i(z_0)-\mathfrak{Q}_i(t)-\mathbf{I}_{\mu_i}(q_j(z_0)-q_j(t))]\exp\,[\mathfrak{Q}_i(\tau)-\mathfrak{Q}_i(z_0)$$

$$-\mathbf{I}_{\mu_i}(q_j(\tau)-q_j(z_0))]|\leqq|\exp\,[\mathfrak{Q}_i(z_0)-\mathfrak{Q}_i(t)-\mathbf{I}_{\mu_i}(q_j(z_0)-q_j(t))]|\,|\exp\,[\mathfrak{Q}_i(\tau)-\mathfrak{Q}_i(z_0)-\mathbf{I}_{\mu_i}(q_j(\tau)-q_j(z_0))]|,\tag{3.26}$$

so that $K_i(\mathscr{P}\cup\mathscr{P}_1)\leqq K_i(\mathscr{P})K_i'(\mathscr{P}_1)$. This shows that the elements of $K_i(\mathscr{P}\cup\mathscr{P}_1)$ $(i=1,\,2,\,\ldots,\,l;\ i\neq j)$ are bounded, and therefore that $\mathscr{D}(z,\,\zeta)$ is a domain.

We next establish the existence and uniqueness of the solution of the integral equation (3.8). To this end, we observe that all known functions in (3.8) are holomorphic and single-valued in every simply-connected subset of $\mathscr{D}(z,\,\zeta)$. *If* $\mathscr{D}(z,\,\zeta)$ *is multiply-connected, we interpret* $\mathscr{D}(z,\,\zeta)$ *to be a Riemann surface on which all known functions of equation* (3.8) *are holomorphic and single-valued.*

Consider the set $\mathfrak{H}(\zeta)$ of all $n\times 1$ vectors $\mathbf{h}(z)$ of functions holomorphic in $\mathscr{D}(z,\,\zeta)$ such that $\lim_{z\to\zeta}\mathbf{h}(z)=\mathbf{O}$, where $z\to\zeta$ along some path $\mathscr{P}$ belonging to the family described above. The set

$\mathfrak{H}(\zeta)$ together with the metric vector

$$d(\mathbf{h}_1, \mathbf{h}_2) = \sup_{t \in \mathscr{P}} |\mathbf{h}_1(t) - \mathbf{h}_2(t)| \tag{3.27}$$

forms a complete [7] metric space.

Let

$$\mathbf{U}^{\nu *}_{jm} = \sup_{\tau \in \mathscr{P}} |U^*_{jm}(\tau)| \tag{3.28}$$

$$\boldsymbol{\gamma}^{\nu}_{jm} = \sup_{t,\, \tau \in \mathscr{P}} |\mathbf{V}^{\nu 1}_{jm}(\tau) + \mathbf{V}^{\nu 2}_{jm}(\tau, t)| \tag{3.29}$$

$$\mathbf{B} = \frac{1}{1-\eta} \left[\sup_{t,\, \tau \in \mathscr{P}} \left| e^{\mathbf{D}_j(\tau) - \mathbf{D}_j(t)} \mathfrak{B}(t) t^{-\eta_1} \right| \right] \tag{3.30}$$

where for example $\mathbf{V}^{\nu k}_{jm}$ $(k = 1,2)$ is the νth column vector in the column of blocks $\mathbf{V}^{k}_{jm}$ $(k = 1,2)$, and t and τ are on $\mathscr{P}$ in the order ζ, t, τ, z. By the analysis of the previous section, all elements of the vectors and matrices defined in (3.28) to (3.30) are bounded. It follows that with $\boldsymbol{\epsilon}^{\nu}_{jm}(t)$ replaced by $\mathbf{h}(t)$, the right of (3.8) is a contraction mapping of a complete metric space into itself, so that the solution to (3.8) exists and is unique.

On substituting the bounds (3.28) to (3.30) into (3.8) and using Lemma 2 of the appendix in [1], we obtain:

THEOREM 3.1: *If, corresponding to a formal vector solution* $\bar{\mathbf{W}}^{\nu}_{\mathrm{jm}}(\mathrm{z})$ *of equation* (2.3), *we can determine a region* $\mathscr{D}(\mathrm{z}, \zeta)$ *satisfying the extreme eigenvalue conditions of Section* 3.2, *then equation* (2.3) *possesses an actual solution vector of functions holomorphic in* $\mathscr{D}(\mathrm{z}, \zeta)$, *given by*

$$\mathbf{W}^{\nu}_{\mathrm{jm}}(\mathrm{z}) = \left[\sum_{\mathrm{k}=0}^{\mathrm{m}-1} \left(\sum_{\mathrm{s}=\nu}^{\mu} \mathbf{U}^{\mathrm{s}}_{\mathrm{jk}} \frac{(\log \mathrm{z})^{\mathrm{s}-\nu}}{(\mathrm{s}-\nu)!} \right) \mathrm{z}^{-\mathrm{k}} + \boldsymbol{\epsilon}^{\nu}_{\mathrm{jm}}(\mathrm{z}) \right] \mathrm{e}^{q_j(\mathrm{z})}, \tag{3.31}$$

in which

$$|\boldsymbol{\epsilon}^{\nu}_{\mathrm{jm}}(\mathrm{z})| \leqq \exp\{\mathbf{B}\mathscr{V}_{\mathscr{P}}(\mathrm{t}^{\eta-1})\} \{\mathbf{U}^{\nu *}_{\mathrm{jm}} \mathscr{V}_{\mathscr{P}}(\mathrm{t}^{\eta_1 - \mathrm{m}}) + \boldsymbol{\gamma}^{\nu}_{\mathrm{jm}} \mathscr{V}_{\mathscr{P}}(\mathrm{t}^{\eta_1 - \mathrm{m} - 1})\} \tag{3.32}$$

for all z *in* $\mathscr{D}(\mathrm{z}, \zeta)$, *where the matrix* $\mathbf{B}$ *and the elements of the vectors* $\mathbf{U}^{\nu *}_{\mathrm{jm}}$ *and* $\boldsymbol{\gamma}^{\nu}_{\mathrm{jm}}$ *are defined by equations* (3.28) *to* (3.30). *The solution* $\mathbf{W}^{\nu}_{\mathrm{jm}}(\mathrm{z})$ *depends on* ζ *and an arbitrary positive integer* m. *The function* $q_j(\mathrm{z})$ *is defined in Theorem* 2.1. *The integers* μ *and* ν *and the vectors* $\mathbf{U}^{\mathrm{s}}_{\mathrm{jk}}$ $(\mathrm{k} = 0, 1, 2, \ldots; \mathrm{s} = 1, 2, \ldots, \mu)$ *are defined as for equation* (2.27). *If each* $\mathbf{J}_{\mu_j} = \mathrm{O}$ $(\mathrm{i} = 1, 2, \ldots, l)$ *then the numbers* η *and* η_1 *can be taken to be zero; if* $\mathbf{J}_{\mu_j} = \mathbf{O}$ *then* η *can be taken to be zero; otherwise* η *and* η_1 *are arbitrary positive numbers less than* 1.

Let us now obtain a norm bound for the vector $\boldsymbol{\epsilon}^{\nu}_{jm}(z)$ in Theorem 3.1. To this end we use the definitions (3.18) to (3.21), and define

$$B = \frac{1}{1-\eta} \sup_{t,\, \tau \in \mathscr{P}} ||e^{\mathbf{D}_j(\tau) - \mathbf{D}_j(t)} \mathfrak{B}(t) t^{-\eta}||, \tag{3.33}$$

$$\mathbf{V}^{\nu *}_{jm} = \sup_{\tau \in \mathscr{P}} ||\mathbf{U}^{\nu *}_{jm}(\tau)||, \qquad \boldsymbol{\gamma}^{\nu}_{jm} = \sup_{t,\, \tau \in \mathscr{P}} ||\mathbf{V}^{\nu 1}_{jm}(\tau) + \mathbf{V}^{\nu 2}_{jm}(\tau, t)||, \tag{3.34}$$

[7] This is easily verified by use of Vitali's theorem.

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where t and τ are on $\mathscr{P}$ in the order ζ, t, τ, z. Substituting these quantities into (3.8) and using [1] Lemma 2, appendix, we obtain

THEOREM 3.2: *A norm bound for the vector* $\boldsymbol{\epsilon}^{\nu}_{jm}(z)$ *in Theorem* 3.1 *is given by*

$$||\boldsymbol{\epsilon}^{\nu}_{jm}(z)|| \leqq \exp\{B\mathscr{V}_{\mathscr{P}}(l^{\eta-1})\}\{U^{\nu *}_{jm}\mathscr{V}_{\mathscr{P}}(l^{\eta_1-m}) + \gamma^{\nu}_{jm}\mathscr{V}_{\mathscr{P}}(l^{\eta_1-m-1})\}, \tag{3.35}$$

where η *and* η_1 *are defined as in Theorem* 3.1 *and* $\mathbf{B}$, $\mathbf{U}^{\nu *}_{jm}$ *and* γ^{ν}_{jm} *are defined in* (3.33) *and* (3.34).

3.5. Error Bounds for the Interior Eigenvalue Case

We shall first establish the existence and uniqueness of the solution of (3.11) by use of norms. We split the vectors $\mathbf{U}^{\nu *}_{jm}(z)$, $\mathbf{V}^{\nu k}_{jm}(k=1,2)$ defined in (3.18) in the manner corresponding to the way (3.7) was split in (3.10) into lower dimensional vectors $\mathbf{U}^{\nu *}_{jms}(z)$, $\mathbf{V}^{\nu k}_{jms}(s=1,2)$. We change ζ to ζ_s if $i \in \mathbf{N}_s$ ($s=1,2$: in the notation of the interior eigenvalue conditions). We then define

$$\psi_{mk}(u, v) = \sup_{t_2 \in \mathscr{P}} ||\mathbf{V}^{\nu 1}_{jmk}(t_2)||\mathscr{V}_{u,v}(l^{\eta_1-m-1}) + \sup_{t_1, t_2 \in \mathscr{P}} \{||\mathbf{V}^{\nu 1}_{jmk}(t_2) + \mathbf{V}^{\nu 2}_{jmk}(t_2, t_1)||\}\mathscr{V}_{u,v}(l^{\eta_1-m-1}) \ (k=1,2) \tag{3.36}$$

where t_1 and t_2 are on $\mathscr{P}$ in the order $\zeta_1, t_1, t_2, \zeta_2$ if $k=1$, or $\zeta_1, t_2, t_1, \zeta_2$ if $k=2$. The fixed number η_1 may be taken to be zero if $\mathbf{J}_{\mu_i}=\mathbf{O}$ ($i=1,2, \ldots, l$) but η_1 is otherwise an arbitrary positive number less than 1.

Similarly, we define

$$a_k = \sup_{t_1, t_2 \in \mathscr{P}} \left|\left| \frac{t_2^{-\eta}}{1-\eta} e^{\mathbf{D}_s(t_2)-\mathbf{D}_s(t_1)} \mathfrak{A}_k(t_1) \right|\right| \qquad \begin{cases} s=1, k=1, 2 \\ s=2, k=3, 4 \end{cases} \tag{3.37}$$

where t_1 and t_2 are again on $\mathscr{P}$ in the order $\zeta_1, t_1, t_2, \zeta_2$ if $s=1$, $k=1, 2$ and in the order $\zeta_1, t_2, t_1, \zeta_2$ if $s=2$, $k=3, 4$. The number η may be taken to be zero if $\mathbf{J}_{\mu_j}=\mathbf{O}$ but otherwise η is an arbitrary positive number less than 1.

LEMMA 3.2: *If under the interior eigenvalue conditions of section* 3.2, $\mathscr{D}(\zeta_1, \zeta_2, \eta)$ *contains at least one finite point* z^* *distinct from* ζ_1 *and* ζ_2, *then* $\mathscr{D}(\zeta_1, \zeta_2, \eta)$ *is a domain.*

PROOF: By assumption there is a path $\mathscr{P}$ in $\mathscr{D}^*$ ($\mathscr{D}^*$ defined in sec. 3.2) connecting ζ_1, z^* and ζ_2 (in that order) and satisfying the three interior eigenvalue conditions. The same path $\mathscr{P}$ con-

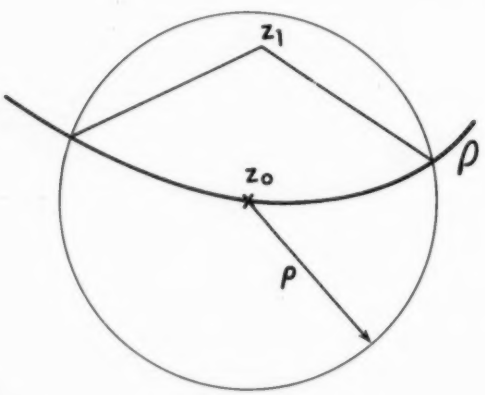

FIGURE 1

nects an arbitrary point z_0 on $\mathscr{P}$ in this manner. By an argument similar to that used in the proof of Lemma 3.1, we can clearly choose a sufficiently small number $\rho > 0$ such that when we alter $\mathscr{P}$ to pass through an arbitrary point z_1 in the interior of the disk $\mathscr{S}(\rho)$ instead then (1) and (3) of the interior eigenvalue conditions remain satisfied. To prove that condition (2) also remains satisfied we need merely show that if $\rho \to 0$ and the altered path $\mathscr{P}'$ in figure 1 tends to $\mathscr{P}$, then

$$\sup_{t,\,\tau \in \mathscr{P}'} \| e^{\mathbf{D}_{j1}(\tau)-\mathbf{D}_{j1}(t)} \mathfrak{a}_1(t) t^{-\eta} \| \to \sup_{t,\,\tau \in \mathscr{P}} \| e^{\mathbf{D}_{j1}(\tau)-\mathbf{D}_{j1}(t)} \mathfrak{a}_1(t) t^{-\eta} \| \tag{3.38}$$

where t and τ are points located on the paths $\mathscr{P}$ and $\mathscr{P}'$ in the order $\zeta_1, t, \tau, \zeta_2$. A convenient way of achieving this is to put

$$\mathbf{\Delta}(t, t_1, \tau, \tau_1) = e^{\mathbf{D}_{j1}(\tau_1)-\mathbf{D}_{j1}(t_1)} \mathfrak{a}_1(t_1) t_1^{-\eta} - e^{\mathbf{D}_{j1}(\tau)-\mathbf{D}_{j1}(t)} \mathfrak{a}_1(t) t^{-\eta} \tag{3.39}$$

where τ and t are located as described above; outside $\mathscr{S}(\rho)$, $t_1 = t$, $\tau_1 = \tau$ while in the interior of $\mathscr{S}(\rho)$ t_1 and τ_1 are arbitrarily located on $\mathscr{P}'$. Let us denote the norm on the left of (3.38) by α', that on the right by α. Then it follows that

$$\alpha' = \alpha + \sup \| \mathbf{\Delta}(t, t_1, \tau, \tau_1) \|, \tag{3.40}$$

where the supremum is taken for $t_1, \tau_1 \in \mathscr{P}'$ and $t, \tau \in \mathscr{P}$. The norm on the right is clearly a continuous function of its elements, and tends to zero as $\rho \to 0$. A similar argument applies to $\mathfrak{a}_2$, $\mathfrak{a}_3$ and $\mathfrak{a}_4$. Moreover the difference $\mathscr{V}_{\mathscr{P}}(t^{\eta-1}) - \mathscr{V}_{\mathscr{P}'}(t^{\eta-1})$ can be made as small as we please by taking ρ sufficiently small. The truth of Lemma 3.2 is now evident.

We again observe that if $\mathscr{D}(\zeta_1, \zeta_2, \eta)$ is simply connected then all known functions appearing in equation (3.11) are single valued in this domain. *If, on the other hand, $\mathscr{D}(\zeta_1, \zeta_2, \eta)$ is multiply connected then we interpret $\mathscr{D}(\zeta_1, \zeta_2, \eta)$ as a Riemann surface on which all known functions appearing in eq.* (3.11) *are single valued.*

Consider the set[8] $\mathfrak{H}(\zeta_1, \zeta_2) = (\mathfrak{H}_1(\zeta_1, \zeta_2), \mathfrak{H}_2(\zeta_1, \zeta_2))^T$ of $n \times 1$ column vectors $\mathbf{h}(z) = (\mathbf{h}_1(z), \mathbf{h}_2(z))^T$ of functions holomorphic in $\mathscr{D}(\zeta_1, \zeta_2, \eta)$, such that $\mathbf{h}_1(z) \in \mathfrak{H}_1(\zeta_1, \zeta_2)$ is $\kappa \times 1$, $\mathbf{h}_2(z) \in \mathfrak{H}_2(\zeta_1, \zeta_2)$ is $(n-\kappa) \times 1$, and all elements of $\mathbf{h}(z)$ are bounded as $z \to \zeta_1$, $z \to \zeta_2$ along any path $\mathscr{P}$ of the family described above. If we take

$$d(x, y) = \begin{matrix} \sup_{t \in \mathscr{P}} \| \mathbf{x}_1(t) - \mathbf{y}_1(t) \| \\ \left(\sup_{t \in \mathscr{P}} \| \mathbf{x}_2(t) - \mathbf{y}_2(t) \| \right) \end{matrix} \tag{3.41}$$

as metric for $\mathfrak{H}$, where $\mathbf{x}_k(t), \mathbf{y}_k(t) \in \mathfrak{H}_k(\zeta_1, \zeta_2)$ $(k = 1, 2)$, then it follows that $\mathfrak{H}(\zeta_1, \zeta_2)$ together with the metric (3.41) form a complete metric space. Substituting (3.36) and (3.37) in (3.11) and using the fact that the eigenvalue of largest modulus of the matrix $\begin{pmatrix} a_1, & a_3 \\ a_2, & a_4 \end{pmatrix} \mathscr{V}_{\mathscr{P}}(t^{\eta-1})$ is $\frac{1}{2}\{a_1 + a_4 + \sqrt{(a_1 - a_4)^2 + 4a_2a_3}\} \mathscr{V}_{\mathscr{P}}(t^{\eta-1})$ and therefore less than 1, we see that the right of (3.11) is a contraction mapping of a complete metric space into itself. Hence the solution of (3.11) exists and is unique.

Following [1] we shall again first obtain a vector bound. We thus define $\psi_{mk}(u, v)$ and $\mathbf{a}_k$ as in (3.36) and (3.37) respectively, except that the norms in these equations are now replaced by absolute values. Thus ψ_{mk} and $\mathbf{a}_k$ are now matrices of absolute values. On substituting these bounds into (3.11) and proceeding as in [1] we obtain

[8] The index T denotes matrix transposition.

THEOREM 3.3: *If, corresponding to the vector* $\tilde{\mathbf{W}}^{\nu}_{jm}(z)$ *(cf., eq.* (2.26)*) we can determine a region* $\mathscr{D}(\zeta_1, \zeta_2, \eta)$ *satisfying the interior eigenvalue conditions of section* 3.2, *then the eq.* (2.3) *possesses an actual solution vector of functions holomorphic in* $\mathscr{D}(\zeta_1, \zeta_2, \eta)$ *and depending on* m, ζ_1 *and* ζ_2, *given by*

$$\begin{pmatrix}\mathbf{W}^{\nu}_{jm1}(z)\\ \mathbf{W}^{\nu}_{jm2}(z)\end{pmatrix}=\left[\sum_{k=0}^{m-1}\left\{\sum_{s=\nu}^{\mu}\begin{pmatrix}U^{s}_{jk1}\\ U^{s}_{jk2}\end{pmatrix}\frac{(\log z)^{s-\nu}}{(s-\nu)!}\right\}z^{-k}+\begin{pmatrix}\epsilon^{\nu}_{jm1}(z)\\ \epsilon^{\nu}_{jm2}(z)\end{pmatrix}\right]e^{q_j(z)} \tag{3.42}$$

Here

$$|\epsilon^{\nu}_{jm1}(z)|\leqq \exp\{\mathbf{a}_2\mathscr{V}_{\zeta_1,z}(t^{\eta-1})\}\psi_{m1}(\zeta_1, z)+\mathbf{C}_1(\zeta_1, z)[\exp\{\mathbf{a}_4\mathscr{V}_{\mathscr{P}}(t^{\eta-1})\}\psi_{m2}(\zeta_1, \zeta_2)+\mathbf{B}] \tag{3.43}$$

and $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, *and* $\mathbf{a}_4$ *are defined by* (3.37), $\Psi_{mk}(u, v)$ $(k=1, 2)$ *are defined by* (3.36) *with norms in these equations replaced by absolute values, and*

$$\mathbf{B}=[\mathbf{I}_2-\mathbf{C}_2\mathbf{C}_1(\zeta_1, \zeta_2)]^{-1}\mathbf{C}_2\times\exp\{\mathbf{a}_1\mathscr{V}_{\mathscr{P}}(t^{\eta-1})\}\Psi_{m1}(\zeta_1, \zeta_2)+\mathbf{C}_1(\zeta_1, \zeta_2)\exp\{\mathbf{a}_4\mathscr{V}_{\mathscr{P}}(t^{\eta-1})\}\Psi_{m2}(\zeta_1, \zeta_2) \tag{3.44}$$

$$\mathbf{C}_1(\zeta_1, z)=\mathbf{a}_1^{-1}[\exp\{\mathbf{a}_1\mathscr{V}_{\zeta_1,\zeta}(t^{\eta-1})\}-\mathbf{I}_1]\mathbf{a}_2;\ \mathbf{C}_2=\mathbf{a}_4^{-1}[\exp\{\mathbf{a}_4\mathscr{V}_{\mathscr{P}}(t^{\eta-1})\}-\mathbf{I}_2]. \tag{3.45}$$

In (3.44) $\mathbf{I}_1$ designates a $\kappa\times\kappa$ unit matrix and $\mathbf{I}_2$ designates an $(n-\kappa)\times(n-\kappa)$ unit matrix, κ being the number of elements in $\mathbf{N}_1$ (sec. 3.2). All variations on the right of (3.43), (3.44), and (3.45) are taken along $\mathscr{P}$. The function $q_j(z)$ is defined as in Theorem 2.1. The vector $\mathbf{U}^{s}_{jk}$ given by Theorem 3.1 is partitioned into lower dimensional vectors as described in section 3.2. The numbers η and η_1 are defined as in Theorem 3.1. The bound (3.43) is valid when every eigenvalue of the matrix $\mathbf{C}_2\mathbf{C}_1(\zeta_1, \zeta_2)$ is less than 1 in magnitude. An exactly similar result holds for $\epsilon^{\nu}_{jm2}(z)$. We also have

THEOREM 3.4: *Let* a_1, a_2, a_3, *and* a_4 *be defined by* (3.37), Ψ_{mk} $(k=1, 2)$ *be defined by* (3.36), *and*

$$\begin{aligned}f(z)&=\mathscr{V}_{\zeta_1, z}(t^{\eta-1}),\\ F(z)&=\frac{a_1a_3}{(a_1+a_1)^2}\left[e^{(a_1+a_4)f(z)}-1-(a_1+a_4)f(z)\right],\end{aligned} \tag{3.46}$$

where the variations are taken along $\mathscr{P}$. *If* $F(\zeta_2)<1$ *then* $1-F(\zeta_2)$ *times the norm of the vector* $\epsilon^{\nu}_{jm1}(z)$ *(cf. eq.* (3.42)*) is bounded by the expression*

$$\begin{aligned}&e^{a_1f(z)}[F(z)+1-F(\zeta_2)]\Psi_{m1}(\zeta_1, z)\\ &+\frac{a_2a_3}{(a_1+a_4)^2}\left[1-e^{-(a_1+a_4)f(z)}\right]\left[e^{(a_2+a_4)f(\zeta_2)}-e^{(a_1+a_4)f(z)}\right]\Psi_{m1}(z, \zeta_2)\\ &+\frac{a_2}{a_1+a_4}e^{a_1f(z)+a_4f(\zeta_2)}\left[1-e^{-(a_1+a_4)f(z)}\right]\Psi_{m2}(\zeta_1, \zeta_2)\end{aligned} \tag{3.47}$$

An exactly similar result holds for $\epsilon^{\nu}_{jm2}(z)$.

4. Example: The Case n = 2

In this section we consider the special system $\mathbf{X}' = z^h \mathfrak{A}(z)\mathbf{X}$ (h an integer), that is

$$\begin{pmatrix} x_{11}' & x_{12}' \\ x_{21}' & x_{22}' \end{pmatrix} = z^h \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \tag{4.1}$$

where for each arbitrary positive integer s

$$a_{ij}(z) = \sum_{k=0}^{s-1} a_{ijk} z^{-k} + a_{ijs}(z) z^{-s} \qquad (i, j = 1, 2), \tag{4.2}$$

and the functions $a_{ijs}(z)$ ($s = 0, 1, 2, \ldots$) are holomorphic in a domain $\mathscr{D}'$ and uniformly bounded on some path $\mathscr{P} \in \mathscr{D}'$ extending to infinity.

We begin by transforming the system (4.1) to canonical form. The flow chart on the following pages indicates the steps involved in transforming the system (4.1) to canonical form. The chart is constructed along the lines of the proof of Turrittin's theorem [3], and it contains all the transformations used in [3] for arbitrary n. The process terminates in all cases.

Box #2 in the chart contains the normalizing transformation, box #8 the exponential transformation, box #19 the shearing transformation, box #27 the zero-inducing transformation, and box #31 the root-equalizing transformation. Box #7 also contains a modified form of the zero-inducing transformation.

The resulting system takes the form

$$\frac{d\mathbf{W}}{dz} = z^r \mathfrak{C}(z)\mathbf{W} = [\mathfrak{Q}'(z) + z^{-2}\mathfrak{B}(z)]\mathbf{W} \tag{4.3}$$

in which r is an integer, and

$$z^r \mathfrak{C}(z) = \left\{ \begin{pmatrix} q_1'(z) & 0 \\ 0 & q_2'(z) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \xi z^{-1} & 0 \end{pmatrix} \right\} + z^{-2} \left\{ \begin{pmatrix} b_{11}(z) & b_{12}(z) \\ b_{21}(z) & b_{22}(z) \end{pmatrix} \right\}; \tag{4.4}$$

$\mathfrak{Q}'(z)$ is the first matrix in braces on the right, $\mathfrak{B}(z)$ is the second, and

$$q_i(z) = \sum_{k=0}^{r} \frac{q_{ik}}{r+1-k} z^{r+1-k} + q_{i,r+1} \log z \qquad (i = 1,2). \tag{4.5}$$

If $r \leqq -2$ then $\mathfrak{Q}'(z) = 0$. If for some integer p in the range $0 \leqq p \leqq r+1$, we have $q_{1p} \neq q_{2p}$, then $\xi = 0$. Otherwise ξ is either 0 or 1. The functions $b_{ij}(z)$ ($i = 1,2$) have the expansion

$$b_{ij}(z) = \sum_{k=0}^{s-1} b_{ijk} z^{-k} + b_{ijs}(z) z^{-s}, \tag{4.6}$$

where the elements of $b_{ijs}(z)$ are holomorphic and uniformly bounded on a path $\mathscr{P}$ extending to infinity in $\mathscr{D}$.

By the analysis of sections 2 to 4, the system (4.3) has a formal solution matrix of the form

$$\tilde{\mathbf{W}}(z) = \tilde{\mathbf{U}}(z) e^{\mathfrak{Q}(z)} \tag{4.7}$$

FLOW CHART ILLUSTRATING TRANSFORMATION OF THE SYSTEM TO CANONICAL FORM WHEN n = 2

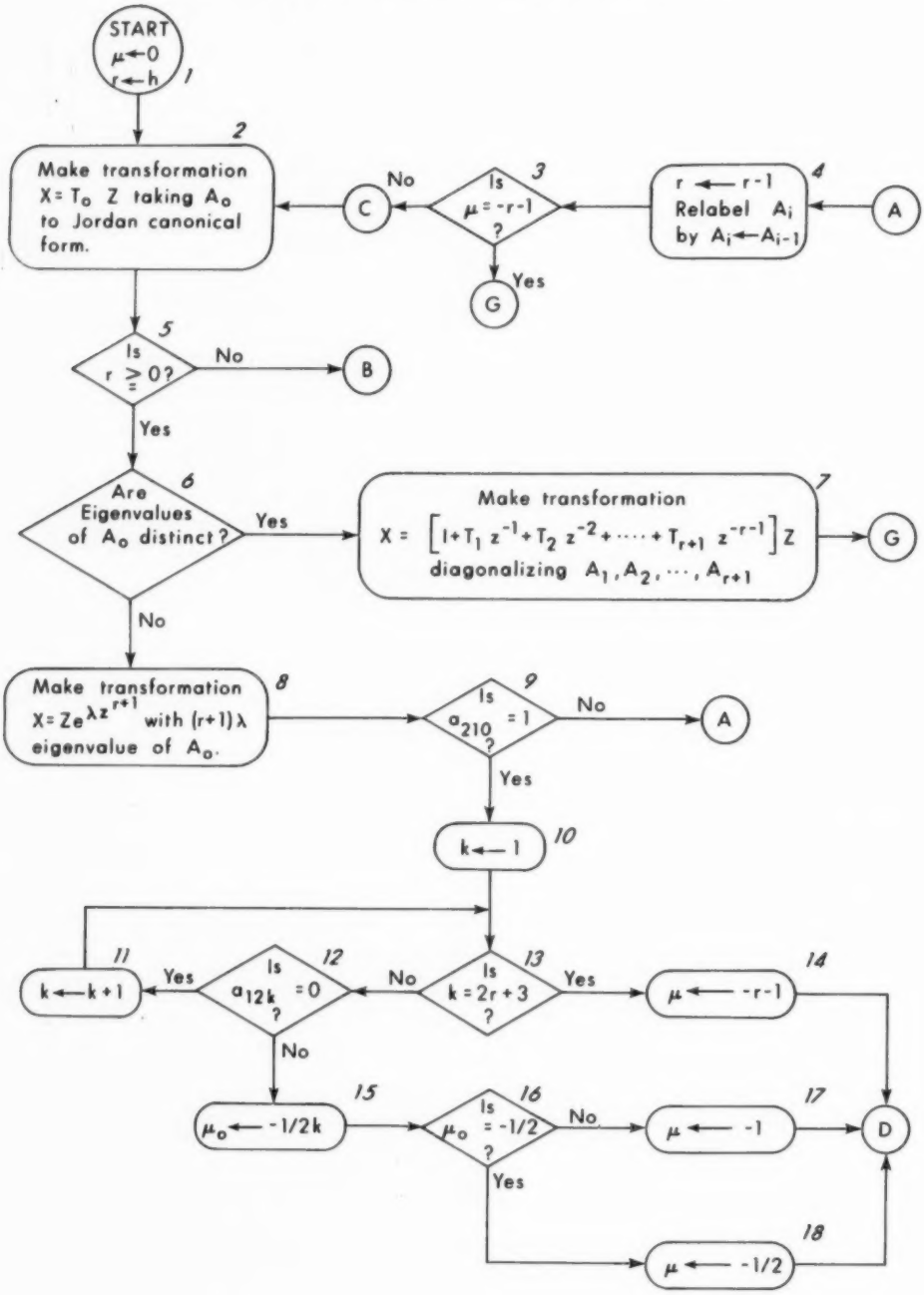

FIGURE 2

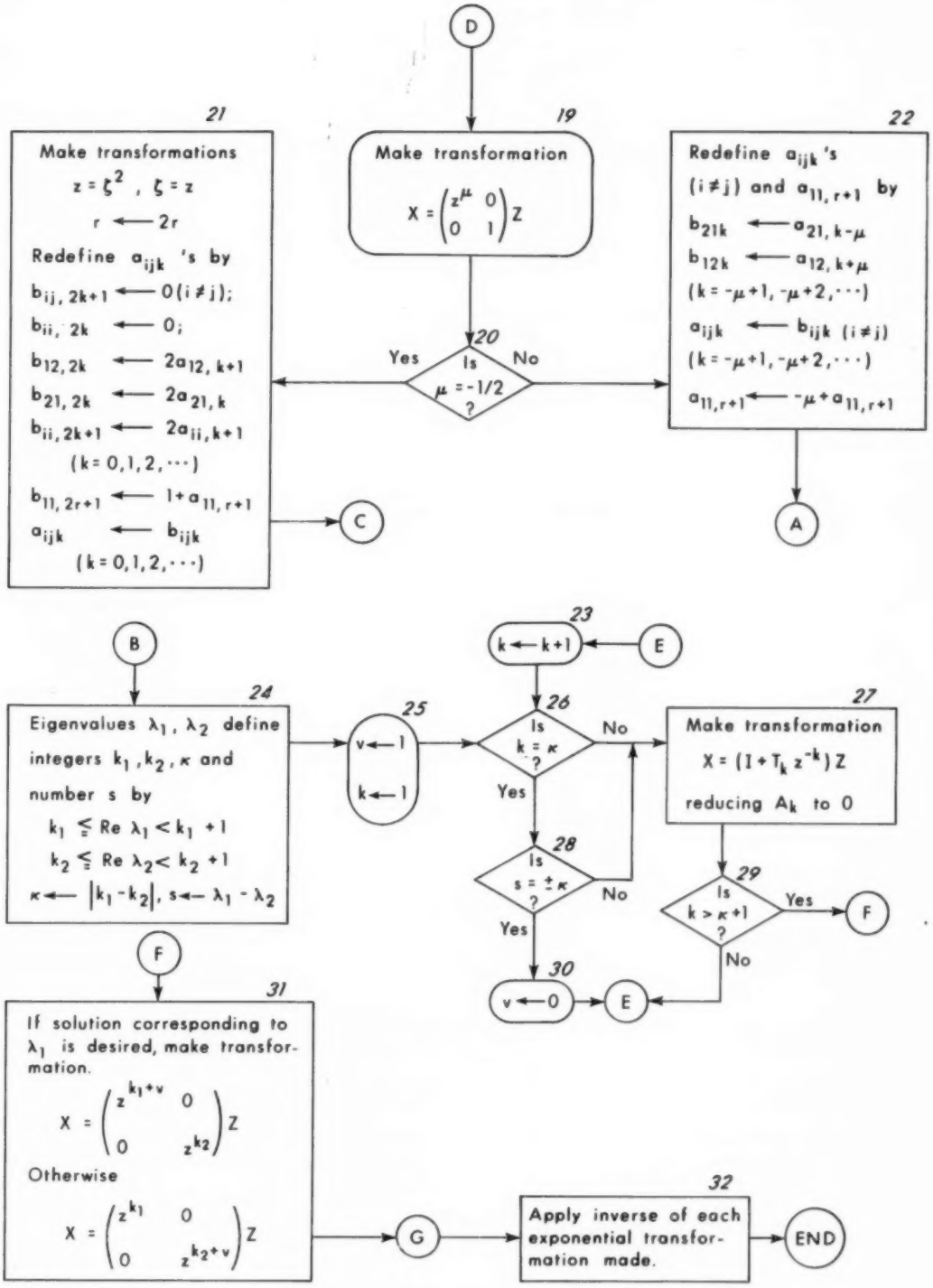

FIGURE 2—Continued

where

$$\Omega(z)=\begin{pmatrix} q_1(z) & 0 \\ 0 & q_2(z) \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ \xi \log z & 0 \end{pmatrix} \tag{4.8}$$

$$\tilde{\mathbf{U}}(z) \triangleq \sum_{k=0}^{\infty} \mathbf{U}_k z^{-k}; \qquad \mathbf{U}_0=\mathbf{I}$$

$$\mathbf{U}_k=\begin{pmatrix} u_{11k} & u_{12k} \\ u_{21k} & u_{22k} \end{pmatrix} k=0,\ 1,\ 2,\ \ldots .$$

Suppose that p is the smallest integer in the range $0 \leqq p \leqq r+1$ such that $q_{1p} \neq q_{2p}$. Then the u_{ijk} are computed from [9]

$$\begin{aligned} &u_{ii0}=1 \qquad (i=1,\ 2) \\ &u_{ijk}=0 \qquad \text{if } i \neq j \text{ and } 0 \leqq k \leqq r+1-p \qquad (i,\ j=1,\ 2) \\ &u_{iik}=-\frac{1}{k}\sum_{s=0}^{k-1}(b_{i1s}u_{1i,\,k-1-s}+b_{i2s}u_{2i,\,k-1-s}) \qquad (i=1,\ 2;\ k \geqq 1) \end{aligned} \tag{4.9}$$

$$u_{ij,\,r+2+k-p}=\frac{1}{q_{ip}-q_{jp}}\sum_{s=p+1}^{r+1} u_{ij,\,r+2+k-s}(q_{js}-q_{is})+(k+1)u_{ij,\,k+1} + \sum_{s=0}^{k}(b_{i1s}u_{ij,\,k-s}+b_{i2s}u_{2j,\,k-s}) \qquad (i \neq j; \qquad k \geqq 0).$$

If, on the other hand, $q_{1p}-q_{2p}=0$ for $p=0,\ 1,\ \ldots,\ r+1$, then $\mathbf{U}_{k+1}$ $(k \geqq 0;\ \mathbf{U}_0=\mathbf{I})$ is computed from

$$(k+1)\mathbf{U}_{k+1}+\mathbf{J}\mathbf{U}_{k+1}-\mathbf{U}_{k+1}\mathbf{J}+\sum_{s=0}^{k}\mathbf{B}_s\mathbf{U}_{k-s}=\mathbf{O} \tag{4.10}$$

with

$$\mathbf{J}=\begin{pmatrix} 0 & 0 \\ \xi & 0 \end{pmatrix}; \qquad \xi=0 \text{ or } 1. \tag{4.11}$$

In either case, whether $\xi=0$ or 1, eq (4.10) with $\mathbf{U}_0=\mathbf{I}$ uniquely determines each $\mathbf{U}_k$, $k \geqq 1$. If $\xi=0$ each column vector can be computed independently of the other. If $\xi=1$ the second column vector in $\mathbf{U}_k$ $(k=0,1,\ \ldots)$ can be computed without computing the first. If, on the other hand, $\xi=1$ and only the first column vector in $\mathbf{U}_k$ $(k=0,1,2\ \ldots)$ is required, the second column vector must also be computed, and the computations for the second column vector must precede the computations for the first.

We shall obtain a bound on the difference between a partial sum of the first formal solution vector and a corresponding actual solution vector. Since the analysis is similar to the preceding extreme eigenvalue case, we only present the main points.

---

[9] The sum $\sum_{s=p+1}^{r+1}$ is to be replaced by zero if $p=r+1$.

In order to define a domain, we choose a nonnegative number a such that if $q_{1p}=q_{2p}$ for $p=0,1,\ldots,r$, then $a=0$ if $|\zeta|=\infty$ and $a=|\zeta|$ if $|\zeta|<\infty$. Let us denote the largest zero of $q'_{21}(z)$ by a_0. If for some $p\in(0,1,\ldots,r)$ $q_{1p}\neq q_{2p}$ then we set $a=|a_0|$ if $|\zeta|=\infty$, while if $|\zeta|<\infty$ we assume that $\zeta\in\mathscr{D}$, ζ does not coincide with a zero of $q'_{21}(z)$ and $|\zeta|\geqq|a_0|$. Let $\mathscr{D}^*=\mathscr{D}-\mathscr{E}_a-S$ where $\mathscr{E}_a$ is the closed disk with center at the origin and radius a, and S is empty or $\{\infty\}$ according to whether or not $\mathfrak{Q}'(z)=\mathbf{O}$. We define a region $\mathscr{D}(z,\zeta)$ to be the union of all points z, other than ζ, such that there is a path $\mathscr{P}$ connecting z and ζ which satisfies the following conditions:

(1) If $\zeta\in\overline{\mathscr{D}^*}-\mathscr{D}^*$, $\mathscr{P}$ lies entirely in $\mathscr{D}^*$ except for ζ;

(2) For given ζ and z, $\exp\{q_{21}(\tau)-q_{21}(t)\}$ is bounded for all points t, τ in the order ζ, t, τ, z on $\mathscr{P}$. If $q_{1p}=q_{2p}$ for $p=0,1,\ldots,r$ and either $q_{1,r+1}\neq q_{2,r+1}$ or $\xi\neq 0$ we also require that τ/t remain bounded;

(3) $\mathscr{V}_{\mathscr{P}}(t^{\eta-1})$ is bounded, where η may be taken to be zero if $\xi=0$, otherwise η is an arbitrary positive number less than 1.

By section 3.3 $\mathscr{D}(z,\zeta)$ is a domain.

If $q_{21}(z)=q_2(z)-q_1(z)\neq 0$ (and hence $\xi=0$) let

$$\mathbf{R}_{11m}(z)=mu_{11m}z^{-m-1}-\sum_{s=0}^{m-1}[b_{11,m-s}(z)u_{11s}+b_{12,m-s}(z)u_{12}]z^{-m-2}$$

$$\begin{aligned}\mathbf{R}_{21m}(z)=&\,q'_{21}(z)u_{21m}z^{-m}-\sum_{k=p+1}^{r+1}(q_{2k-q1k})z^{r-k}u_{21m}z^{-m}\\&-\sum_{\mu=m+p+1}^{m+r+1}\Bigg[\sum_{s=\mu+1-m}^{\min(\mu,r+1)}(q_{2s}-q_{1s})u_{21,\mu-s}+(\mu-r-1)u_{21,\mu-r-1}\\&+\sum_{s=0}^{\mu-r-2}(b_{11s}u_{11,\mu-r-2-s}+b_{12s}u_{21,\mu-r-2-s})\Bigg]z^{r-\mu}-\sum_{s=0}^{m-1}[b_{11,m-s}(z)u_{11s}+b_{12,m-s}(z)u_{21s}]z^{-m-2},\end{aligned}\tag{4.12}$$

where p is the smallest nonnegative integer such that $q_{2p}\neq q_{1p}$. On the other hand, if $q_2(z)=q_1(z)$ then

$$\begin{aligned}\begin{pmatrix}\mathbf{R}_{11m}(z)\\ \mathbf{R}_{21m}(z)\end{pmatrix}=&\left\{\left[-\begin{pmatrix}u_{11m}&u_{12m}\\ u_{21m}&u_{22m}\end{pmatrix}\begin{pmatrix}0&0\\ \xi&0\end{pmatrix}+\begin{pmatrix}m&0\\ \xi&m\end{pmatrix}\begin{pmatrix}u_{11m}&u_{12m}\\ u_{21m}&u_{22m}\end{pmatrix}\right]z^{-m-1}\right.\\&\left.+\sum_{s=0}^{m-1}\left[\begin{pmatrix}b_{11,m-s}(z)&b_{12,m-s}(z)\\ b_{21,m-s}(z)&b_{22,m-s}(z)\end{pmatrix}\begin{pmatrix}u_{11s}&u_{12s}\\ u_{21s}&u_{22s}\end{pmatrix}z^{-m-2}\right]\right\}\begin{pmatrix}1\\ \xi\log z\end{pmatrix}.\end{aligned}\tag{4.13}$$

The integral equation for the difference between an actual solution vector and a partial sum of a formal solution vector thus takes the form

$$\boldsymbol{\epsilon}_{1m}(z)=\int_{\zeta}^{z}\exp[\mathfrak{Q}(z)-\mathfrak{Q}(t)-\mathbf{I}(q_1(z)-q_1(t))]\{t^{-2}\mathfrak{B}(t)\boldsymbol{\epsilon}_{1m}(t)-\mathbf{R}_{1m}(t)\}dt\tag{4.14}$$

where the path of integration satisfies the above conditions, and $\mathbf{R}_{1m}(z)=(\mathbf{R}_{11m}(z),\mathbf{R}_{21m}(z))^T$.

We define nonnegative numbers u^*_{11m} and u^*_{21m} to be the suprema when $\tau \in \mathscr{P}$ of the respective expressions

$$\left|\frac{(u_{11m}+\xi u_{12m}\log\tau)\tau^{-m}-(u_{11m}+\xi u_{21m}\log\zeta)\zeta^{-m}}{\tau^{\eta_1-m}-\zeta^{\eta_1-m}}\right|$$

$$\left|\frac{(u_{21m}+\xi u_{22m}\log\tau)\tau^{-m}e^{q_{21}(\tau)-q_{21}(\zeta)}-\left\{u_{21m}+\xi\left[\log\frac{\tau}{\zeta}(u_{11m}+u_{21m}\log\zeta)+u_{22m}\log\zeta\right]\right\}\zeta^{-m}}{\tau^{\eta_1-m}-\zeta^{\eta_1-m}}\right| \tag{4.15}$$

where η_1 may be taken to be zero if $\xi=0$ but otherwise η_1 is an arbitrary positive number less than 1.

Similarly with t_1 and t_2 points on $\mathscr{P}$ in the order ζ, t_1, t_2, z we define

$$\gamma_{11m}=\sup_{t\in\mathscr{P}}\left|\frac{t^{-\eta_1}}{m+1-\eta_1}\sum_{s=0}^{m-1}\{b_{11,\,m-s}(t)[u_{11s}+\xi\log t\ u_{12s}]+b_{12,\,m-s}(t)[u_{21s}+\xi\log t\ u_{22s}]\}\right|$$

$$\gamma_{21m}=\sup_{t_1,\,t_2\in\mathscr{P}}\left|\frac{p_{21}(t_2)/q'_{21}(t_2)-e^{q_{21}(t_2)-q_{21}(\zeta)}\,p_{21}(\zeta)/q'_{21}(\zeta)}{t_2^{\eta_1-m}-\zeta^{\eta_1-m}}+\frac{t^{-\eta_1}}{m+1-\eta_1}\left[e^{q_{21}(t_2)-q_{21}(t_1)}\left\{-\frac{d}{dt_1}\left(\frac{p_{21}(t_1)}{q'_{21}(t_1)}\right)\right\}t_1^{m+2}\right.\right.$$

$$\left.\left.+\sum_{s=0}^{m-1}\sum_{k=1}^{2}\left\{\left[b_{2k,\,m-s}(t_1)+\xi\log\left(\frac{t_2}{t_1}\right)\ b_{1k,\,m-s}(t_1)\right][u_{k1s}+\xi\log t_1 u_{k2s}]\right\}+t_1^{m+2}\frac{d}{dt_1}\left(\frac{p_{21}(t_1)}{q'_{21}(t_1)}\right)\right]\right| \tag{4.16}$$

where

$$p_{21m}(t)=\mathbf{R}_{21m}(t)-q'_{21}(t)u_{21m}t^{-m}+\sum_{s=0}^{m-1}[b_{11,\,m-s}(t)u_{11s}+b_{12,\,m-s}(t)u_{21s}]t^{-m-2} \tag{4.17}$$

with $\mathbf{R}_{21m}$ given by (4.12), and t_1, t_2 points on $\mathscr{P}$ in the order ζ, t_1, t_2, z. Note that $p_{21}(t)=0$ if the smallest integer p such that $q_{2p}\neq q_{1p}$ is $r+1$.

We define

$$(1-\eta)a_k=\sup_{t\in\mathscr{P}}\left|t^{-\eta}b_{1k}(t)\right|$$

$$(1-\eta)a_s=\sup_{t_1,\,t_2\in\mathscr{P}}\left|t_1^{-\eta}\left[b_{2k}(t_1)+\xi\log\left(\frac{t_2}{t_1}\right)\ b_{1k}(t_1)\right]e^{q_{21}(t_2)-q_{21}(t_1)}\right| \tag{4.18}$$

$$(k=1,2\ ;\ s=k+2).$$

In (4.18) t_1 and t_2 are defined as in (4.16), and η may be taken zero if ξ is zero; otherwise η is an arbitrary positive number less than 1.

Summarizing we have

THEOREM 4.1: *If, corresponding to the formal vector solution* $\widetilde{\mathbf{W}}_1(z)$ *of equation* (4.3), *we can determine a domain* $\mathcal{D}(z, \zeta)$ *satisfying the conditions above, then the equation* (4.3) *possesses an actual solution vector of functions holomorphic in* $\mathcal{D}(z, \zeta)$, *given by*

$$\mathbf{W}_{1m}(z) = \begin{bmatrix} \mathbf{W}_{11m}(z) \\ \mathbf{W}_{21m}(z) \end{bmatrix} = \left[\sum_{k=0}^{m-1} \left\{ \begin{pmatrix} u_{11k} \\ u_{21k} \end{pmatrix} + \xi \log z \begin{pmatrix} u_{12k} \\ u_{22k} \end{pmatrix} \right\} z^{-k} + \begin{pmatrix} \epsilon_{11m}(z) \\ \epsilon_{21m}(z) \end{pmatrix} \right] e^{q_1(z)} \tag{4.19}$$

where

$$\begin{pmatrix} |\epsilon_{11m}(z)| \\ |\epsilon_{21m}(z)| \end{pmatrix} \leqq \exp \left\{ \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} \mathcal{V}_{\mathcal{P}} (t^{\eta-1}) \right\} \times \left\{ \begin{matrix} u^*_{11m} \mathcal{V}_{\mathcal{P}} (t^{\eta_1 - m}) + \gamma_{11m} \mathcal{V}_{\mathcal{P}} (t^{\eta_1 - m - 1}) \\ u^*_{21m} \mathcal{V}_{\mathcal{P}} (t^{\eta_1 - m}) + \gamma_{21m} \mathcal{V}_{\mathcal{P}} (t^{\eta_1 - m - 1}) \end{matrix} \right\} \tag{4.20}$$

for all z *in* $\mathcal{D}(z, \zeta)$. *The solution* $\mathbf{W}_{1m}(z)$ *depends on* ζ *and an arbitrary positive integer* m. *The function* $q_1(z)$ *is defined by* (4.5); *the numbers* u^*_{11m}, γ_{11m} *and* a_1, a_2, a_3, a_4 *are defined by* (4.15), (4.16), *and* (4.18). *If* $\xi = 0$, *both* η *and* η_1 *may be taken to be zero; otherwise* η *and* η_1 *are arbitrary positive numbers less than* 1.

The right side of (4.20) may be explicitly evaluated by use of

$$\exp \begin{Bmatrix} \alpha & \beta \\ \gamma & \delta \end{Bmatrix} = \left[\frac{\sinh \chi}{\chi} \begin{pmatrix} \frac{\alpha - \delta}{2} & \beta \\ \gamma & \frac{\delta - \alpha}{2} \end{pmatrix} + \cosh \chi \, \mathbf{I} \right] e^{1/2(\alpha + \delta)};$$

$$\chi = \tfrac{1}{2} \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}.$$

5. The Choice of Paths

The conditions on the paths of integration in this paper are somewhat weaker and define a larger domain of validity for the asymptotic expansions than the conditions given in [1]. The paths which minimize the error bounds under the present conditions are not known. In application, we suggest that paths similar to those of [5] be tried; the possible loss in accuracy is offset by the greater ease of evaluation of the variations.

Often the error bound can be appreciably reduced by the following simple procedure. Consider for example, the distinct eigenvalue case 1. Let $\mathbf{T}_0$ be a matrix such that $\mathbf{T}_0^{-1}\mathbf{A}_0\mathbf{T}_0 = \mathbf{\Lambda}$ is diagonal. Then $(\mathbf{T}_0\mathbf{D})^{-1}\mathbf{A}_0(\mathbf{T}_0\mathbf{D}) = \mathbf{\Lambda}$, where $\mathbf{D}$ is an arbitrary nonsingular diagonal matrix. We choose $\mathbf{D}$ so that the least upper bound of the off-diagonal elements in the resulting coefficient matrix is minimized on the path $\mathcal{P}$. This applies also to the more general case of this paper, except that here we may destroy the similarity of the canonical form to the Jordan canonical form. If we want to preserve the canonical form then for every Jordan leading subblock of $\mathfrak{C}(z)$ of the form

$$\sum_{k=0}^{r} q_{ik} z^{r-k} \mathbf{1} + q_{i,\,r+1} \mathbf{J} \log z$$

where $\mathbf{J}$ has a full set of units in its first lower diagonal and is zero elsewhere, we must make the corresponding elements of $\mathbf{D}$ equal.

On inspecting the proof of Theorem 2.1, [1], we observe that the undeterminable diagonal elements of $\mathbf{T}_k (k = 1, 2, \ldots, r+1)$ were chosen to be zero. By introducing nonzero diagonal elements we may again be able to improve on the error bounds. A similar observation applies to the zero-inducing transformations used in [3].

6. Summary

The results of this paper and the preceding paper extend the work of Olver [5] for the second-order equation to a system of first-order equations. In the majority of applications it is expected that the eigenvalues of the lead coefficient matrix $\mathbf{A}_0$ are distinct. The analysis for this case is considerably simpler and different from the analysis for the case when the eigenvalues for the lead coefficient matrix are not distinct, and this is the main reason that the presentation has been split into two parts.

In all cases the procedure was to transform (1.1) to canonical form, to obtain formal solutions for the transformed system, and to obtain actual solutions such that the formal solutions are asymptotic expansions of the actual solutions of the transformed system, as $z \to \infty$ in some sector of the complex plane. Two cases arise when the eigenvalues of $\mathbf{A}_0$ are distinct: the extreme eigenvalue case and the interior eigenvalue case. They are characterized geometrically by the relative location of the eigenvalues of $\mathbf{A}_0$ in the complex plane. The main distinction between these two cases is that in the extreme eigenvalue case it is possible to express the error vector—that is, the difference between an actual solution vector and a partial sum of a formal solution vector—by use of a single Volterra vector integral equation, while in the interior eigenvalue case it is necessary to use a simultaneous pair of Volterra vector integral equations. As a consequence, the error bounds are considerably sharper in the extreme eigenvalue case.

When the coefficient matrix in (1.1) is regular for all sufficiently large z and when the eigenvalues of $\mathbf{A}_0$ are distinct, it is always possible to choose end-points of integration in the Volterra vector integral equations at infinity. This is no longer the case when the eigenvalues of $\mathbf{A}_0$ are not distinct. Further, in this latter case we can no longer always give a simple geometric characterization of extreme and interior eigenvalue cases in terms of the relative location of the eigenvalues of $\mathbf{A}_0$. We can, nevertheless, always express the error vector by at most a simultaneous pair of Volterra vector integral equations.

The case $n=2$ is likely to be the most important in applications. The complete solution for this case is given in detail, including a full statement of the transformation to canonical form.

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7. References

[1] Stenger, F., Error bounds for asymptotic solutions of differential equations. I. The distinct eigenvalue case, J. Res. NBS **70B** (Math. and Math. Phys.) No. 3. 167 (1966).

[2] Hokuhara, M., Sur les points singuliers des equations differentielles lineares, II of the Fac. of Sci., Hokkaido Imp. Univ., Ser. I, **5**, 123–166, (1937).

[3] Turrittin, H. L., Convergent solutions of ordinary differential equations in the neighborhood of an irregular singular point, Acta. Math. **93**, 27–66, (1955).

[4] Trjitzinsky, W. J., Analytic theory of linear differential equations, Act. Math. **62**, 167–226, (1933).

[5] Olver, F. W. J., On the asymptotic solution of second order differential equations having an irregular singularity of rank 1, with an application to Whittaker functions, J. Soc. Indust. Appl. Math. Num. Anal. Ser. B, 225–243 (1965).

[6] Thorne, R. C., Asymptotic formulae for solutions of linear second order differential equations with large parameter, Australian Math. Soc. **1**, 439–464, (1960).

[7] Stenger, F., Error bounds for solutions of differential equations, Ph.D. thesis, University of Alberta, Edmonton (1965).

[8] Collatz, L., Funktionalanalysis und Numerische Mathematik, Springer-Verlag (1964).

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On the Approximation of Functions of Several Variables[1]

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The purpose of this note is to point out how a certain type of approximation to functions of one real variable gives rise to similar approximations to functions of several variables. Information on the rapidity of convergence in the one dimensional case, yields at once corresponding information for the multidimensional case.

Key Word: Approximations, convergence, multidimensional, polynomials, functions, variables, Hermite-Fejér, Bernstein, Chebyshev.

1. The purpose of this note is to point out how a certain type of approximation to functions of one real variable, gives rise to similar approximations to functions of several variables. In fact, we show how information on the rapidity of convergence of the approximation in the one dimensional case, yields at once corresponding information for the multidimensional case.

2. Let $-\infty < a < b < \infty$, and for $n = 1, 2, \ldots$, let $c_0^{(n)}, c_1^{(n)}, \ldots, c_n^{(n)}$ be points of $[a, b]$, and $K_0^{(n)}(x), K_1^{(n)}(x), \ldots, K_n^{(n)}(x)$ polynomials which are ≥ 0 throughout $[a, b]$, and such that

$$\sum_{j=0}^{n} K_j^{(n)}(x) \equiv 1.$$

We set, finally, for every real function f, continuous in $[a, b]$,

$$P_n(f, x) \equiv \sum_{j=0}^{n} f(c_j^{(n)}) K_j^{(n)}(x) \qquad (n = 1, 2, \ldots). \tag{1}$$

3. The purpose of constructing such polynomials $P_n(f, x)$ is to obtain polynomial approximations to f. Here are two examples.

I. Let $a = 0, b = 1$, and for $n = 1, 2, \ldots$, let $c_j^{(n)} = j/n$, $K_j^{(n)}(x) \equiv \binom{n}{j} x^j (1-x)^{n-j}$ $(j = 0, 1, \ldots, n)$. Then $\Sigma_{j=0}^{n} K_j^{(n)}(x) \equiv 1$ $(n = 1, 2, \ldots)$. If f is any real function, continuous in $[0, 1]$, then for $n = 1. 2, \ldots$.

$$P_n(f, x) \equiv \Sigma_{j=0}^{n} f(c_j^{(n)}) K_j^{(n)}(x) \equiv \Sigma_{j=0}^{n} f(j/n) \binom{n}{j} x^j (1-x)^{n-j} \tag{2}$$

is the Bernstein polynomial [1] of order n of f. If, furthermore, $\omega(\delta)$ is the modulus of continuity of f in $[0, 1]$, then ([7, 10]) the polynomial (2) satisfies

$$\max_{0 \leq x \leq 1} |f(x) - P_n(f, x)| \leq (5/4)\omega(n^{-1/2}) \qquad (n = 1, 2, \ldots).$$

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II. Let $a=-1$, $b=1$, and for $n=1, 2, \ldots$, let $c_j^{(n)}=\cos\left[\frac{2j+1}{2(n+1)}\pi\right]$,

$$K_j^{(n)}(x)\equiv(1-c_j^{(n)}x)[T_{n+1}(x)/\{(n+1)(x-c_j^{(n)})\}]^2 \qquad (j=0, 1, \ldots, n),$$

where $T_{n+1}(x)\equiv 2^n\Pi_{k=0}^{n}(x-c_k^{(n)})$ is the $(n+1)$th degree Chebyshev polynomial of the first kind satisfying $T_{n+1}(\cos\theta)\equiv\cos[(n+1)\theta]$. If f is any real function continuous in $[-1, 1]$, then

$$P_n(f, x)\equiv\Sigma_{j=0}^{n}f(c_j^{(n)})K_j^{(n)}(x)\equiv\Sigma_{j=0}^{n}f(c_j^{(n)})(1-c_j^{(n)}x)[T_{n+1}(x)/\{(n+1)(x-c_j^{(n)})\}]^2$$

$(n=1, 2, \ldots)$ are the well-known [3] Hermite-Fejér polynomials converging uniformly to f in $[-1, 1]$. Again $\Sigma_{j=0}^{n}K_j^{(n)}(x)\equiv 1$ $(n=1, 2, \ldots)$. If f is a real function, satisfying throughout $[-1,1]$

$$|f(v)-f(u)|\leq\lambda|v-u|$$

where λ is a positive constant, then [11] for $n=1, 2, \ldots$

$$\max_{-1\leq x\leq 1}|f(x)-P_n(f, x)|<4\lambda\pi(n+1)^{-1}[\alpha+\log(n+1)]$$

where $\alpha=\frac{1}{2}+C-\log 2=0.384\ldots$, C being Euler's constant. Furthermore, if f is a real function, continuous in $[-1, 1]$, and if $\omega(\delta)$ is the modulus of continuity of f there, then for $n=1,2, \ldots$ we have [11]

$$\max_{-1\leq x\leq 1}|f(x)-P_n(f, x)|\leq[2+4\pi+\eta_n]\omega\left(\frac{\log(n+1)}{n+1}\right),$$

where η_n depends on n only and $\eta_n\to 0$ as $n\to\infty$.

4. THEOREM 1. *Assume the hypotheses and notation of section 2. Assume also that if* f *is a real function satisfying for some positive constant* λ, *throughout* [a, b], $|f(v)-f(u)|\leq\lambda|v-u|$, *then for* $n=1, 2, \ldots$

$$\max_{a\leq x\leq b}|f(x)-P_n(f, x)|<a_{\lambda, n}$$

where $a_{\lambda, n}$ *depends on* λ *and* n *only. Let* $f(x_1, x_2, \ldots, x_p)$ $(p\geq 2)$ *be a real function, defined on the cube* C: $a\leq x_k\leq b$, $k=1, 2, \ldots, p$. *Suppose that for* $r=1, 2, \ldots, p$, λ_r *is a positive number such that throughout C*

$$|f(x_1, x_2, \ldots, x_{r-1}, v, x_{r+1}, \ldots, x_p)-f(x_1, x_2, \ldots, x_{r-1}, u, x_{r+1}, \ldots, x_p)|\leq\lambda_r|v-u|. \quad (3)$$

Let $n_1, n_2, \ldots, n_p$ *be arbitrary positive integers, and set*

$$P_{n_1, n_2, \ldots, n_p}(f, x_1, x_2, \ldots, x_p)\equiv\sum_{h_1=0}^{n_1}\cdots\sum_{h_p=0}^{n_p}f(c_{h_1}^{(n_1)}, \ldots, c_{h_p}^{(n_p)})K_{h_1}^{(n_1)}(x_1)\cdots K_{h_p}^{(n_p)}(x_p). \quad (4)$$

Then throughout C:

$$|f(x_1, x_2, \ldots x_p)-P_{n_1, n_2, \ldots, n_p}(f, x_1, x_2, \ldots, x_p)|<\sum_{r=1}^{p}a_{\lambda_r, n_r}.$$

PROOF. Observe that $\sum_{h_1=0}^{n_1} \cdots \sum_{h_p=0}^{n_p} K_{h_1}^{(n_1)}(x_1) \cdots K_{h_p}^{(n_p)}(x_p) \equiv 1$.

Hence, throughout C,

$$f(x_1, \ldots, x_p) - P_{n_1, n_2, \ldots, n_p}(f, x_1, x_2, \ldots, x_p)$$

$$= \sum_{h_1=0}^{n_1} \cdots \sum_{h_p=0}^{n_p} [f(x_1, \ldots, x_p) - f(c_{h_1}^{(n_1)}, \ldots, c_{h_p}^{(n_p)})] K_{h_1}^{(n_1)}(x_1) \cdots K_{h_p}^{(n_p)}(x_p)$$

$$= \sum_{h_1=0}^{n_1} \cdots \sum_{h_p=0}^{n_p} \left\{ \sum_{r=1}^{p} [f(c_{h_1}^{(n_1)}, \ldots, c_{h_{r-1}}^{(n_{r-1})}, x_r, \ldots, x_p) - f(c_{h_1}^{(n_1)}, \ldots, c_{h_r}^{(n_r)}, x_{r+1}, \ldots, x_p)] \right\}$$

$$\cdot \prod_{s=1}^{p} K_{h_s}^{(n_s)}(x_s)$$

$$= \sum_{r=1}^{p} \sum_{\substack{h_q=0,1,\ldots,n_q \\ q=1,2,\ldots,p,\, q\neq r}} \left\{ \sum_{h_r=0}^{n_r} [f(c_{h_1}^{(n_1)}, \ldots, c_{h_{r-1}}^{(n_{r-1})}, x_r, \ldots, x_p) - f(c_{h_1}^{(n_1)}, \ldots, c_{h_r}^{(n_r)}, x_{r+1}, \ldots, x_p)] \right.$$

$$\left. \cdot K_{h_r}^{(n_r)}(x_r) \right\} \prod_{\substack{s=1 \\ s\neq r}}^{p} K_{h_s}^{(n_s)}(x_s).$$

$(f(c_{h_1}^{(n_1)}, \ldots, c_{h_{r-1}}^{(n_{r-1})}, x_r, \ldots, x_p)$ means $f(x_1, \ldots, x_p)$ if $r=1$, and

$(f(c_{h_1}^{(n_1)}, \ldots, c_{h_r}^{(n_r)}, x_{r+1}, \ldots, x_p)$ means $f(c_{h_1}^{(n_1)}, \ldots, c_{h_p}^{(n_p)})$ if $r=p$.

Thus, throughout C,

$$|f(x_1, \ldots, x_p) - P_{n_1, n_2, \ldots, n_p}(f, x_1, \ldots, x_p)|$$

$$\leq \sum_{r=1}^{p} \sum_{\substack{h_q=0,1,\ldots,n_q \\ q=1,2,\ldots,p,\, q\neq r}} \left| f(c_{h_1}^{(n_1)}, \ldots, c_{h_{r-1}}^{(n_{r-1})}, x_r, \ldots, x_p) - \sum_{h_r=0}^{n_r} f(c_{h_1}^{(n_1)}, \ldots, c_{h_r}^{(n_r)}, x_{r+1}, \ldots, x_p) \right.$$

$$\left. \cdot K_{h_r}^{(n_r)}(x_r) \right| \prod_{\substack{s=1 \\ s\neq r}}^{p} K_{h_s}^{(n_s)}(x_s)$$

$$< \sum_{r=1}^{p} \sum_{\substack{h_q=0,1,\ldots,n_q \\ q=1,2,\ldots,p,\, q\neq r}} a_{\lambda_r, n_r} \prod_{\substack{s=1 \\ s\neq r}}^{p} K_{h_s}^{(n_s)}(x_s) = \sum_{r=1}^{p} a_{\lambda_r, n_r}.$$

5. EXAMPLE [11]. Let $f(x_1, x_2, \ldots, x_p)$ $(p \geq 2)$ be a real function defined on the cube C: $-1 \leq x_k \leq 1$, $k=1,2, \ldots, p$. Suppose that for $r=1,2, \ldots, p$ we have, for some positive constant λ_r and throughout C, the inequality (3). For $j=0,1, \ldots, n$; $n=1,2, \ldots$, let $c_j^{(n)}$ and $K_j^{(n)}(x)$ be as in II, section 3. If $n_1, n_2, \ldots, n_p$ are positive integers and $P_{n_1, n_2, \ldots, n_p}(f, x_1, \ldots, x_p)$ is defined by (4), then by Theorem 1 we have, throughout C,

$$|f(x_1, \ldots, x_p) - P_{n_1, \ldots, n_p}(f, x_1, \ldots, x_p)| < \sum_{r=1}^{p} 4\lambda_r \pi (n_r+1)^{-1} [\alpha + \log(n_r+1)].$$

6. Similarly to Theorem 1, one can prove the following

THEOREM 2. *Assume the hypotheses and notation of Section 2. Assume also that for* $n=1,2,\ldots$, α_n, $\beta_n(\beta_n \leq b-a)$ *are numbers such that if* f *is a real function, continuous in* [a, b], *with modulus of continuity* $\omega(\delta)$ *there, then*

$$\max_{a \leq x \leq b} |f(x)-P_n(f, x)| \leq \alpha_n \omega(\beta_n).$$

Let $f(x_1, x_2, \ldots, x_p)$ $(p \geq 2)$ *be a real function, continuous in the cube* $C: a \leq x_k \leq b$, $k=1,2,\ldots,p$. *For every* $\delta \epsilon [0, b-a]$ *and every* $r(=1,2,\ldots,p)$, *let*

$$\omega_r(\delta)=\max|f(x_1, \ldots, x_{r-1}, v, x_{r+1}, \ldots, x_p)-f(x_1, \ldots, x_{r-1}, u, x_{r+1}, \ldots, x_p)| \tag{5}$$

where the x_j, u, *and* v *vary in* [a, b] *with* $0 \leq v-u \leq \delta$. *Let* $n_1, n_2, \ldots, n_p$ *be arbitrary positive integers. Then with the notation* (4), *we have throughout* C,

$$|f(x_1, \ldots, x_p)-P_{n_1, n_2, \ldots, n_p}(f, x_1, \ldots, x_p)| \leq \sum_{r=1}^{p} \alpha_{n_r} \omega_r(\beta_{n_r}). \tag{6}$$

7. EXAMPLE. Let $f(x_1, \ldots, x_p)$ $(p \geq 2)$ be a real function, continuous in the cube $C: 0 \leq x_k \leq 1$, $k=1,2,\ldots,p$. For every $\delta \epsilon [0,1]$ and every $r(=1,2,\ldots,p)$ let $\omega_r(\delta)$ be as in Theorem 2 (with $a=0$, $b=1$). Let $n_1, \ldots, n_p$ be arbitrary positive integers. Then by Theorem 2, and by I, section 3, we have throughout C:

$$\left| f(x_1, \ldots, x_p)-\sum_{h_1=0}^{n_1} \cdots \sum_{h_p=0}^{n_p} f\left(\frac{h_1}{n_1}, \ldots, \frac{h_p}{n_p}\right) \binom{n_1}{h_1} \cdots \binom{n_p}{h_p} x_1^{h_1} \cdots x_p^{h_p}(1-x_1)^{n_1-h_1} \right.$$

$$\left. \cdots (1-x_p)^{n_p-h_p} \right| \leq \sum_{r=1}^{p} (5/4)\omega_r(n_r^{-1/2}).$$

8. We consider now an analog of the situation considered in Theorems 1 and 2, sums being replaced by integrals.

9. Let $-\infty < a < b < \infty$, and for $n=1,2,\ldots$ let $K_n(x, t)$ be a real function which is ≥ 0 and continuous in the square $a \leq x \leq b$, $a \leq t \leq b$, and which satisfies for every $x \epsilon [a, b]$,

$$\int_a^b K_n(x, t)dt=1.$$

For every real function f, continuous in $[a, b]$, set

$$P_n(f, x) \equiv \int_a^b f(t)K_n(x, t)dt \qquad (n=1,2,\ldots). \tag{7}$$

10. Such $P_n(f, x)$ are constructed again, like their counterparts (1), in order to obtain approximations to f. Here are a few examples of such $P_n(f, x)$ prominent in Analysis.

I. Let $a=-\pi$, $b=\pi$ and for every real x, t, let

$$K_n(x, t)=\frac{1}{\pi}\left[-\frac{1}{2}+\sum_{j=0}^{n-1} \frac{n-j}{n} \cos\{j(t-x)\}\right] \qquad (n=1,2,\ldots).$$

If $t-x$ is not an integral multiple of 2π, then

$$K_n(x, t)=\sin^2\left\{\frac{n}{2}(t-x)\right\} \Big/ \left[2n\pi \sin^2\left\{\frac{1}{2}(t-x)\right\}\right] \qquad (n=1,2,\ldots).$$

Also the properties in the first sentence of section 9 hold. For every real function f continuous in $(-\infty, \infty)$ and of period 2π, and for $n=1,2, \ldots, P_n(f, x)$ of (7) is in the present case the arithmetic mean

$$\sigma_n(f, x) \equiv n^{-1} \sum_{j=0}^{n-1} S_j(x),$$

where

$$S_0(x) \equiv a_0/2, \qquad S_j(x) \equiv (a_0/2) + \sum_{k=0}^{j} [a_k \cos(kx) + b_k \sin(kx)] \qquad (j=1,2, \ldots),$$

$$(a_0/2) + \sum_{k=0}^{\infty} [a_k \cos(kx) + b_k \sin(kx)]$$

being the Fourier series of f, and by a classical theorem of Fejér [2] $\sigma_n(f, x)$ converges uniformly to $f(x)$ in $(-\infty, \infty)$. Furthermore, suppose that a real function f(of period 2π) satisfies throughout the real line, for some constant λ,

$$|f(v) - f(u)| \leq \lambda |v-u|.$$

Then by a theorem of S. N. Bernstein ([6], p. 61; [9], p. 162) for every $n > 1$, $P_n(f, x) \equiv \sigma_n(f, x)$ satisfies

$$\max_{-\infty < x < \infty} |f(x) - P_n(f, x)| \leq C_0 \lambda \log n/n,$$

C_0 being an absolute constant.

II. Let $a = -\pi$, $b = \pi$, and for every real x, t, let

$$K_n(x, t) = \frac{3}{2\pi n(2n^2+1)} \left[-n + 2 \sum_{j=0}^{n-1} (n-j) \cos\{j(t-x)\} \right]^2 \qquad (n=1,2, \ldots).$$

If $t-x$ is not an integral multiple of 2π, then

$$K_n(x, t) = \frac{3}{2\pi n(2n^2+1)} \left[\frac{\sin\left\{\frac{n}{2}(t-x)\right\}}{\sin\left(\frac{t-x}{2}\right)} \right]^4 \qquad (n=1,2, \ldots).$$

The properties in the first sentence of section 9 hold. For every real function f, continuous in $(-\infty, \infty)$ and of period 2π, and for $n=1,2, \ldots, P_n(f, x)$ of (7) is now a trigonometric polynomial introduced by Jackson ([4, 5, 6]). If f is a real function of period 2π, satisfying for every real u, v

$$|f(v) - f(u)| \leq \lambda |v-u|$$

λ being a constant, then by a classical theorem of Jackson ([4, 5, 6]) this particular $P_n(f, x)$ satisfies

$$\max_{-\infty < x < \infty} |f(x) - P_n(f, x)| \leq (c\lambda/n) \qquad (n=1, 2, \ldots),$$

c being an absolute constant.

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III. Let $a=-\pi$, $b=\pi$, and for every real x, t, let

$$K_n(x, t)=\frac{(2n)!!}{2\pi(2n-1)!!}\cos^{2n}\left(\frac{t-x}{2}\right) \qquad (n=1, 2, \ldots).$$

Here, $m!!$ denotes $2\cdot4\cdot6\cdot\ldots m$ for every positive even m, and $1\cdot3\cdot5\cdot\ldots m$ for every positive odd m. Again the properties in the first sentence of section 9 hold. For every real function f, continuous and of period 2π in $(-\infty, \infty)$ and for $n=1,2, \ldots$, $P_n(f, x)$ of (7) is in the present case a trigonometric polynomial introduced by de la Valée-Poussin [12]. If $\omega(\delta)$ $(0\leq\delta<\infty)$ is the modulus of continuity of such an f, then ([8], [9]) for $n=1,2, \ldots$

$$\max_{-\infty<x<\infty}|f(x)-P_n(f, x)|\leq 3\omega(n^{-1/2}).$$

11. An analog of Theorem 1 is

THEOREM 3. *Assume the hypotheses and notation of section 9. Assume also that if* f *is a real function satisfying for some positive constant* λ, *throughout* [a, b],

$$|f(v)-f(u)|\leq\lambda|v-u|,$$

then for $n=1,2, \ldots$ $\max_{a\leq x\leq b}|f(x)-P_n(f, x)|<a_{\lambda, n}$

where $a_{\lambda, n}$ *depends on* λ *and* n *only. Let* $f(x_1, x_2, \ldots, x_p)$ $(p\geq 2)$ *be a real function defined in the cube* C: $a\leq x_k\leq b$, $k=1,2, \ldots, p$. *Suppose that for* $r=1,2, \ldots, p$, λ_r *is a positive number such that throughout* C

$$|f(x_1, x_2, \ldots, x_{r-1}, v, x_{r+1}, \ldots, x_p)-f(x_1, x_2, \ldots, x_{r-1}, u, x_{r+1}, \ldots, x_p)|\leq\lambda_r|v-u|. \qquad (8)$$

Let $n_1, n_2, \ldots, n_p$ *be arbitrary positive integers, and set*

$$P_{n_1, n_2, \ldots, n_p}(f, x_1, \ldots, x_p)\equiv\int_a^b\ldots\int_a^b f(t_1, \ldots, t_p)K_{n_1}(x_1, t_1)\ldots K_{n_p}(x_p, t_p)dt_1\ldots dt_p. \qquad (9)$$

Then throughout C:

$$|f(x_1, \ldots, x_p)-P_{n_1, n_2, \ldots, n_p}(f, x_1, \ldots, x_p)|<\sum_{r=1}^{p}a_{\lambda_r, n_r}.$$

PROOF. Observe that throughout C,

$$\int_a^b\ldots\int_a^b K_{n_1}(x_1, t_1)\ldots K_{n_p}(x_p, t_p)dt_1\ldots dt_p=1.$$

Hence, throughout C,

$$f(x_1, \ldots x_p)-P_{n_1, n_2, \ldots, n_p}(f, x_1, \ldots, x_p)$$

$$=\int_a^b\ldots\int_a^b\left[f(x_1, \ldots x_p)-f(t_1, \ldots, t_p)\right]K_{n_1}(x_1, t_1)\ldots K_{n_p}(x_p, t_p)dt_1\ldots dt_p$$

$$=\int_a^b\ldots\int_a^b\left[\sum_{r=1}^{p}f(t_1, \ldots, t_{r-1}, x_r, \ldots, x_p)-f(t_1, \ldots, t_r, x_{r+1}, \ldots, x_p)\right]K_{n_1}(x_1, t_1)$$

$$\ldots K_{n_p}(x_p, t_p)dt_1\ldots dt_p$$

$$= \sum_{r=1}^{p} \int_a^b \cdots \int_a^b \left\{ \int_a^b [f(t_1, \ldots, t_{r-1}, x_r, \ldots, x_p) \right.$$

$$\left. - f(t_1, \ldots, t_r, x_{r+1}, \ldots, x_p)] K_{n_r}(x_r, t_r) dt_r \right\} \prod_{\substack{s=1 \\ s \neq r}}^{p} K_{n_s}(x_s, t_s) dt_1 \ldots dt_{r-1} dt_{r+1} \ldots dt_p.$$

($f(t_1, \ldots, t_{r-1}, x_r, \ldots, x_p)$ means $f(x_1, \ldots, x_p)$ if $r=1$, and $f(t_1, \ldots, t_r, x_{r+1}, \ldots, x_p)$ means $f(t_1, \ldots, t_p)$ if $r=p$). Thus, throughout C,

$$|f(x_1, \ldots, x_p) - P_{n_1, n_2, \ldots, n_p}(f, x_1, \ldots, x_p)| \leq \sum_{r=1}^{p} \int_a^b \cdots \int_a^b \left| f(t_1, \ldots, t_{r-1}, x_r, \ldots, x_p) \right.$$

$$\left. - \int_a^b f(t_1, \ldots, t_r, x_{r+1}, \ldots, x_p) K_{n_r}(x_r, t_r) dt_r \right| \prod_{\substack{s=1 \\ s \neq r}}^{p} K_{n_s}(x_s, t_s) dt_1 \ldots dt_{r-1} dt_{r+1} \ldots dt_p$$

$$< \sum_{r=1}^{p} \int_a^b \cdots \int_a^b a_{\lambda_r, n_r} \prod_{\substack{s=1 \\ s \neq r}}^{p} K_{n_s}(x_s, t_s) dt_1 \ldots dt_{r-1} dt_{r+1} \ldots dt_p = \sum_{r=1}^{p} a_{\lambda_r, n_r}.$$

Similarly, one can prove the following analog of Theorem 2:

THEOREM 4. *Assume the hypotheses and notation of section 9. Assume also that for* $n = 1,2, \ldots, \alpha_n, \beta_n (\beta_n \leq b - a)$ *are numbers such that if* f *is a real function, continuous in* [a, b], *with modulus of continuity* $\omega(\delta)$ *there, then*

$$\max_{a \leq x \leq b} |f(x) - P_n(f, x)| \leq \alpha_n \omega(\beta_n).$$

Let $f(x_1, \ldots, x_p)$ $(p \geq 2)$ *be a real function, continuous in the cube* C: $a \leq x_k \leq b$, $k = 1,2, \ldots, p$. *For every* $\delta \epsilon [0, b-a]$ *and every* $r (= 1,2, \ldots, p)$ *let* $\omega_r(\delta)$ *be as in Theorem 2. Let* $n_1, \ldots, n_p$ *be arbitrary positive integers. Then with the notation* (9), *we have* (6) *throughout* C.

12. The last two theorems can obviously be modified in the following way. For $n = 1,2, \ldots$ let $K_n(x, t)$ be a real function which throughout the plane $-\infty < x < \infty$, $-\infty < t < \infty$ is ≥ 0, continuous, and of period 2π with respect to x and to t, and such that for every real x,

$$\int_{-\pi}^{\pi} K_n(x, t) dt = 1.$$

For every real function f, continuous and of period 2π in $(-\infty, \infty)$, set

$$P_n(f, x) \equiv \int_{-\pi}^{\pi} f(t) K_n(x, t) dt \qquad (n = 1,2, \ldots).$$

Let $f(x_1, \ldots, x_p)$ $(p \geq 2)$ be a real function, continuous and of period 2π with respect to each x_j in the (real) Euclidean p-space E_p. Let $n_1, n_2, \ldots, n_p$ be positive integers, and set

$$P_{n_1, \ldots, n_p}(f, x_1, \ldots, x_p) \equiv \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(t_1, \ldots, t_p) K_{n_1}(x_1, t_1) \ldots K_{n_p}(x_p, t_p) dt_1 \ldots dt_p. \quad (10)$$

A. Suppose that if f is a real function of period 2π, satisfying for some constant λ, throughout $(-\infty, \infty)$, $|f(v)-f(u)| \leq \lambda|v-u|$, then for $n=1,2, \ldots$

$$\max_{-\infty<x<\infty} |f(x)-P_n(f, x)| \leq a_{\lambda, n}$$

where $a_{\lambda, n}$ depends on λ and n only. Suppose that for $r=1,2, \ldots, p$, λ_r is a number such that (8) holds everywhere in E_p. Then throughout E_p we have

$$|f(x_1, \ldots, x_p)-P_{n_1, \ldots, n_p}(f, x_1, \ldots, x_p)| \leq \sum_{r=1}^{p} a_{\lambda_r, n_r}.$$

For example, if for $n=1,2, \ldots, K_n(x, t)$ is as in example II of section 10, then throughout E_p we have

$$\left|f(x_1, \ldots, x_p)-P_{n_1, \ldots, n_p}(f, x_1, \ldots, x_p)\right| \leq c \sum_{r=1}^{p} \lambda_r/n_r,$$

c being the absolute constant mentioned there.

B. Assume that for $n=1,2, \ldots, \alpha_n$ and β_n are numbers such that if f is a real function, continuous and of period 2π in $(-\infty, \infty)$, with modulus of continuity $\omega(\delta)(0 \leq \delta < \infty)$, then

$$\max_{-\infty<x<\infty} |f(x)-P_n(f, x)| \leq \alpha_n \omega(\beta_n).$$

For every $\delta \geq 0$ and for $r=1,2, \ldots, p$ let $\omega_r(\delta)$ be given by (5) where now the x_j, u and v vary in $(-\infty, \infty)$, subject to $0 \leq v-u \leq \delta$. Then (with the notation (10)), (6) holds throughout E_p. For example, if for $n=1,2, \ldots, K_n(x, t)$ is as in example III of section 10, then throughout E_p we have

$$\left|f(x_1, \ldots, x_p)-P_{n_1, \ldots, n_p}(f, x_1, \ldots, x_p)\right| \leq 3 \sum_{r=1}^{p} \omega_r(n_r^{-1/2}).$$

References

[1] S. N. Bernstein, Démonstration du théorème de Weierstrass, fondeé sur le calcul des probabilités, Commun. Soc. Math. Kharkow (2), **13,** 1–2 (1912–13).

[2] L. Fejér, Untersuchungen über Fouriersche Reihen, Math. Ann. **58,** 51–69 (1904).

[3] L. Fejér, Über Interpolation, Göttinger Nachrichten, pp. 66–91 (1916).

[4] D. Jackson, Über die Genauigkeit der Annäherung stetiger Funktionen durch ganze rationale Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung. Dissertation, Göttingen (1911).

[5] D. Jackson, On approximation by trigonometric sums and polynomials, Trans. Amer. Math. Soc. **13,** 491–515 (1912).

[6] D. Jackson, The theory of approximation, New York (1930).

[7] G. G. Lorentz, Bernstein Polynomials, Toronto (1953).

[8] I. P. Natanson, Some estimates connected with the singular integral of de la Vallée-Poussin (Russian), Doklady Akad. Nauk SSSR **45,** 290–293 (1944).

[9] I. P. Natanson, Constructive function theory, Volume **I,** (English translation) New York (1964).

[10] T. Popoviciu, Sur l'approximation des fonctions convexes d'ordre supérieur, Mathematica (Cluj) **10,** 49–54 (1935).

[11] O. Shisha and B. Mond, The rapidity of convergence of the Hermite-Fejér approximation to functions of one or several variables, Proc. Amer. Math. Soc. **16,** 1269–1276 (1965).

[12] Ch. J. de la Vallée-Poussin, Sur l'approximation des fonctions d'une variable réelle et de leurs dérivées par les polynômes et des suites limitées de Fourier, Bull. de l'Acad. Royale de Belgique, 193–254 (1908).

(Paper 70B3–182)

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Finding a Rank-Maximizing Matrix Block

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(May 11, 1966)

An algorithm is developed for the following problem: Given three matrices of respective dimensions $s\times s$, $s\times t$, and $t\times s$, to find a $t\times t$ matrix such that the $(s+t)\times(s+t)$ matrix formed from the four blocks has maximum rank.

Key words: Matrix, block, rank, algorithm.

Let A, B, C be matrices (over any field) of respective dimensions $s\times s$, $s\times t$, $t\times s$. We shall show how to find a $t\times t$ matrix X such that

$$M(X)=\begin{bmatrix} A & B \\ C & X \end{bmatrix}$$

has maximum possible rank $\rho(M(X))$, or equivalently has minimum possible deficiency

$$\delta(M(X))=s+t-\rho(M(X)).$$

Let β' and γ' be the respective ranks of the matrices

$$[A \quad B],\ \begin{bmatrix} A \\ C \end{bmatrix}.$$

It is readily seen that for any choice of X,

$$\delta(M(X))\geq \max\,(s-\beta',\ s-\gamma'). \tag{1}$$

We will show that X can be chosen so that *equality* holds in (1). Thus

$$\min_X \delta(M(X))=\max\,(s-\beta',\ s-\gamma'). \tag{2}$$

For the special case $\beta'=\gamma'=s$, this was proved by Pearl.[1]

Denote the rank of A by α. Find nonsingular $s\times s$ matrices U and V which bring A into Smith normal form, i.e.,

$$UAV=I_\alpha \dot{+} 0_{s-\alpha}.$$

$M(X)$ will have the same rank and deficiency as

$$M_1(X)=\begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} M(X) \begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} UAV & UB \\ CV & X \end{bmatrix},$$

which we repartition as

$$M_1(X)=\begin{bmatrix} I_\alpha & 0 & B_1 \\ 0 & 0_{s-\alpha} & B_2 \\ C_1 & C_2 & X \end{bmatrix}$$

where $$[C_1C_2]=CV,\ \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}=UB.$$

$M_1(X)$, in turn, has the same rank and deficiency as

$$M_2(X)=\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -C_1 & 0 & I \end{bmatrix} M_1(X) \begin{bmatrix} I & 0 & -B_1 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$=\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & B_2 \\ 0 & C_2 & X-C_1B_1 \end{bmatrix}.$$

Let β and γ be the respective ranks of B_2 and C_2. Since $[A\ B]$ has the same rank as

$$U[A\ B](V\dot{+}I_t)\begin{bmatrix} I & 0 & -B_1 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I_\alpha & 0 & 0 \\ 0 & 0 & B_2 \end{bmatrix}$$

we see that $\beta'=\alpha+\beta$, and similarly $\gamma'=\alpha+\gamma$.

Find a nonsingular $(s-\alpha)\times(s-\alpha)$ matrix U_2 and a nonsingular $t\times t$ matrix V_2 which together bring B_2 into Smith normal form, i.e.,

$$U_2B_2V_2=\begin{bmatrix} I_\beta & 0 \\ 0 & 0 \end{bmatrix}.$$

[1] Martin Pearl, On Normal *EPr* Matrices, Mich. Math. J. **8**, 33–37 (1961), (Theorem 2).

Then $M_2(X)$ has the same rank and deficiency as

$$M_3(X)=\begin{bmatrix} I & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & I \end{bmatrix} M_2(X) \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & V_2 \end{bmatrix}$$

$$=\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & U_2B_2V_2 \\ 0 & C_2 & (X-C_1B_1)V_2 \end{bmatrix}.$$

Next find a nonsingular $t \times t$ matrix U_3 and a nonsingular $(s-\alpha)\times(s-\alpha)$ matrix V_3 which together bring C_2 into Smith normal form, i.e.,

$$U_3C_2V_3=\begin{bmatrix} I_\gamma & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $M_3(X)$ has the same rank and deficiency as

$$M_4(X)=\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & U_3 \end{bmatrix} M_3(X) \begin{bmatrix} I & 0 & 0 \\ 0 & V_3 & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$=\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & U_2B_2V_2 \\ 0 & U_3C_2V_3 & U_3(X-C_1B_1)V_2 \end{bmatrix},$$

which we repartition as

$$M_4(X)=\begin{bmatrix} I_\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_\beta & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I_\gamma & 0 & Y_1 & Y_2 \\ 0 & 0 & 0 & Y_3 & Y_4 \end{bmatrix}, \tag{3}$$

where $$U_3(X-C_1B_1)V_2=\begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}.$$

At this point β and γ are known. Assume without loss of generality that $\beta \leq \gamma$. Choose $Y_1=0$, $Y_2=0$, $Y_3=0$, and choose the $(t-\gamma)\times(t-\beta)$ matrix $Y_4=[I_{t-\gamma}0]$. On the one hand, this choice yields

$$\rho(M(X))=\rho(M_4(X))=\alpha+\gamma+\beta+(t-\gamma)$$

$$=\alpha+t+\min(\beta,\gamma)=t+\min(\beta',\gamma',),$$

so that equality holds in (1). On the other hand the choice is achieved by setting

$$X=C_1B_1+U_3^{-1}\begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{t-\gamma} & 0 \end{bmatrix} V_2^{-1}. \tag{4}$$

If $\gamma<\beta$, then (4) can be replaced by

$$X=C_1B_1+U_3^{-1}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & I_{t-\beta} \end{bmatrix} V_2^{-1}; \tag{5}$$

i.e., $Y_1=0$, $Y_2=0$, $Y_3=0$, $Y_4=[0\ I_{t-\beta}]$.

The main computational labor involved comes (a) in bringing A into Smith normal form, (b) in determining V_2^{-1} and β, and (c) in determining U_3^{-1} and γ. Step (a) can be carried out by performing elementary row operations to "sweep out" A to upper triangular form (the product of these operators' matrices is U), and then performing elementary column operations (the product of whose matrices is V) to "sweep out" the resulting matrix into diagonal form $D_\alpha+0_{s-\alpha}$; it is not necessary to normalize D_α to I_α. As for step (b), we can take V_2 as the product of the matrices of elementary column operations used to sweep out B_2 into lower triangular form; each elementary matrix is trivially invertible so that V_2^{-1} can be built up during the sweep-out process, and β can be read off at the end of the process. Similarly for step (c).

(Paper 70B3–183)

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On Certain Discrete Inequalities and Their Continuous Analogs *

Allen M. Pfeffer [1]

(May 24, 1966)

The purpose of this paper is to find inequalities between the L^2-norms of a function and its kth and mth derivatives.

Key Words: Inequalities, norms, Wirtinger.

1. Introduction

In 1955 [2] [2] Fan, Taussky, and Todd discovered discrete analogs of certain integral inequalities involving functions and their derivatives. They considered inequalities of the Wirtinger type: $\int_0^{2\pi} x(t)^2 dt \leq \int_0^{2\pi} x'(t)^2 dt$, where x has period 2π and $\int_0^{2\pi} x(t)dt = 0$. By taking limits they were able to derive continuous inequalities by matrix techniques and avoided the differential equations of the calculus of variations. We have attempted to generalize the techniques of [2] to polynomials in the derivatives of x. We have also considered analogs of inequalities of Müller [5] and Redheffer [7] and have used these inequalities to derive a necessary and sufficient condition on ordered pairs of numbers so that the first number is the square norm of the kth derivative of some periodic function and the second number is the square norm of the mth derivative of the same periodic function. This last result is the L_2 analog of a result of Kolmogoroff [4] on the uniform norm.

2. General Technique

The following is the basic technique of Fan, Taussky, and Todd [2] and will be used to derive most of our inequalities. Let S be a real symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and let $v_1, v_2, \ldots, v_n$ be the corresponding linearly independent eigenvectors:

$$Sv_i = \lambda_i v_i (i = 1,2, \ldots, n).$$

If P is a polynomial with real coefficients, then $P(S)$ is a real symmetric $n \times n$ matrix and we know that $P(S)$ has eigenvalues $P(\lambda_1), P(\lambda_2), \ldots, P(\lambda_n)$ with corresponding eigenvectors $v_1, v_2, \ldots, v_n$ so that $P(S)v_i = P(\lambda_i)v_i$ $(i = 1,2, \ldots, n)$.

By a well-known property of symmetric matrices

$$\min_{[1 \leq i \leq n\{P(\lambda_i)\}]} \cdot (x, x) \leq (x, P(S)x) \leq \max_{[1 \leq i \leq n\{P(\lambda_i)\}]} \cdot (x, x)$$

for every n-vector x, where $(x, y) = x^T y$ for n-vectors x and y.

Now we must have

$$\inf_{[\lambda_n \leq t \leq \lambda_1\{P(t)\}]} \leq \min_{[1 \leq n\{P(\lambda_i)\}]}$$

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[2] Figures in brackets indicate the literature references at the end of this paper.

and

$$\sup_{\lambda_n \le t \le \lambda_1\{P(t)\}} \le \max_{1 \le i \le n\{P(\lambda_i)\}},$$

hence

$$\inf_{\lambda_n \le t \le \lambda_1\{P(t)\}} \cdot (x, x) \le (x, P(S)x) \le \sup_{\lambda_n \le t \le \lambda_1\{P(t)\}} \cdot (x, x)$$

for every n-vector x.

We note that equality is possible in the last expression if and only if $P(\lambda_i) = \sup_{\lambda_n \le t \le \lambda_1\{P(t)\}}$ for some i between 1 and n or $P(\lambda_i) = \inf_{\lambda_n \le t \le \lambda_1\{P(t)\}}$ for some i between 1 and n. If equality does occur then the set of extremal vectors must be the space spanned by the eigenvectors corresponding to λ_i.

For reasons of simplicity we will only consider vectors with real components and real symmetric matrices, in all cases the extension to complex vectors and Hermitian matrices will be apparent.

3. Periodic Boundary Conditions

Because of the absence of troublesome boundary conditions the periodic case is easiest to handle. We may periodically extend any n-vector $x = (x_1, x_2, \ldots, x_n)^T$ by setting $x_{i+rn} = x_i$ for $i = 1, 2, \ldots, n$ and r any integer.

DEFINITION: If x is a periodically extended n-vector then for $m = 0, 1, 2, \ldots$

$$x^{(m)} = (\Delta^m x_1, \Delta^m x_2, \ldots, \Delta^m x_n)^T,$$

where

$$\Delta^m x_i = \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} x_{i-[m/2]+r} \qquad (i = 1, 2, \ldots, n).$$

We call $x^{(m)}$ the mth difference of the n-vector x.

It is clear that $x^{(r+s)} = (x^{(r)})^{(s)}$ for $r, s = 0, 1, 2, \ldots$.

The following lemma, which is crucial for our inequalities, was proved by Fan-Taussky-Todd [2] for $m = 0, 1, 2$.

LEMMA 1. *If* x *is a periodically extended* n*-vector, then* $(x^{(m)}, x^{(m)}) = (x, P^m x)$ $(m = 0, 1, 2, \ldots)$, *where* P *is the* $n \times n$ *symmetric circulant*

$$\begin{bmatrix} 2 & -1 & 0 & . & . & . & -1 \\ -1 & 2 & -1 & . & . & . & 0 \\ . & . & . & & & & . \\ . & . & . & & & & . \\ . & . & . & & & & . \\ -1 & 0 & . & . & . & -1 & 2 \end{bmatrix}$$

PROOF:

$$Px = \begin{bmatrix} -x_n + 2x_1 - x_2 \\ \cdot \\ \cdot \\ \cdot \\ -x_1 + 2x_n - x_{n-1} \end{bmatrix} = -x^{(2)}, \text{ hence } Px^{(j)} = -x^{(j+2)}.$$

By partial summation

$$(x, x^{(2)}) = \sum_{k=1}^{n} x_k(x_{k-1} - 2x_k + x_{k+1}) = -\sum_{k=1}^{n} (x_{k+1} - x_k)^2 = -(x^{(1)}, x^{(1)}),$$

yielding

$$(x^{(j)}, Px^{(j)}) = (x^{(j+1)}, x^{(j+1)}).$$

Thus if m is even we have

$$(x^{(m)}, x^{(m)}) = (P^{\frac{m}{2}}x, P^{\frac{m}{2}}x) = (x, P^m x),$$

while if m is odd

$$(x^{(m)}, x^{(m)}) = (x^{(m-1)}, Px^{(m-1)}) = (P^{\frac{m-1}{2}}x, P \cdot P^{\frac{m-1}{2}}x) = (x, P^m x).$$

Rutherford [8] has shown that P has eigenvalues $4 \sin^2\left(\frac{k\pi}{n}\right)$ $(k = 1, 2, \ldots, n)$, hence $\lambda_n = 0$, $\lambda_{n-1} = \lambda_{n-2} = 4 \sin^2\left(\frac{\pi}{n}\right), \ldots, \lambda_1 = 4 \sin^2\left(\left[\frac{n}{2}\right]\frac{\pi}{n}\right)$. The eigenvector corresponding to λ_n is $(1, 1, \ldots, 1)^T$.

At this point we have two alternatives open to us. If F is a polynomial with real coefficients then we may use the method that we have described in section 2 or else we may also add the auxiliary condition $\sum_{k=1}^{n} x_i = 0$, which is equivalent to the requirement $(x, e) = 0$, where $e = (1, 1, \ldots, 1)^T$. Since e is the eigenvector corresponding to $\lambda_n = 0$, the auxiliary orthogonality condition implies that we need only take our maximum and minimum over $\{F(\lambda_1), F(\lambda_2), \ldots, F(\lambda_{n-1})\}$.

The following example should illustrate the differences in the two approaches.

Let $P(t) = t^m$ for $m \geq 1$. Since $P(t)$ is increasing for nonnegative t and since $0 = \lambda_n < \lambda_{n-1} \leq \ldots \leq \lambda_1$, we have $1 \leq \overset{\min}{i} \leq n\{P(\lambda_i)\} = 0$ and $1 \leq \overset{\min}{i} \leq n-1\{P(\lambda_i)\} = P(\lambda_{n-1}) = 4^m \sin^{2m}\left(\frac{\pi}{n}\right)$.

Thus if x is a periodically extended n-vector then we have the trivial inequality $(x^{(m)}, x^{(m)}) \geq 0(x, x) = 0$; however if we add the auxiliary condition $\sum_{i=1}^{n} x_i = 0$, then we obtain $(x^{(m)}, x^{(m)}) \geq 4^m \sin^{2m}\left(\frac{\pi}{n}\right)(x, x)$.

We note that the above inequalities are best possible and we include a discussion of the possibility of equality. For brevity this will be the only case where we discuss equality in the discrete case.

If $(x^{(m)}, x^{(m)}) = 0$, then x must be in the subspace spanned by the eigenvectors corresponding to $\lambda_n = 0$. Thus $x = a \cdot (1, 1, \ldots, 1)^T$, where a is a real number.

If $\sum_{i=1}^{n} x_i = 0$ and $(x^{(m)}, x^{(m)}) = 4^m \sin^{2m}\left(\frac{\pi}{n}\right) \cdot (x, x)$, then x must be in the subspace spanned by the eigenvectors corresponding to $\lambda_{n-1} = \lambda_{n-2} = 4 \sin^2\left(\frac{\pi}{n}\right)$. It is known, [2], that this subspace is spanned by the vectors $u = (u_1, u_2, \ldots, u_n)^T$ and $w = (w_1, w_2, \ldots, w_n)^T$, where $u_j = \cos\left(\frac{2\pi j}{n}\right)$ and $w_j = \sin\left(\frac{2\pi j}{n}\right)$. Thus $x = a \cdot u + b \cdot w$, where a and b are real numbers.

Hence we have established the following extension of Theorem 10 of [2].

THEOREM 1. *If* x *is a periodically extended* n*-vector and if* $\sum_{i=1}^{n} x_i = 0$, *then*

$$(x^{(m)}, x^{(m)}) \geq 4^m \sin^{2m}\left(\frac{\pi}{n}\right) \cdot (x, x).$$

Equality holds if and only if x *is the periodic extension of a vector of the form* $a \cdot u + b \cdot w$.

COROLLARY 1.1. *If* $x(t) \in C^m[a, b]$, $x(t)$ *has period* $b - a$, *and* $\int_a^b x(t)dt = 0$, *then*

$$\int_a^b \{x^{(m)}(t)\}^2 dt \geq \left(\frac{2\pi}{b-a}\right)^{2m} \int_a^b \{x(t)\}^2 dt.$$

PROOF:

If we let $\Delta = \frac{b-a}{n+1}$ and $y_i = x(t_i) - \frac{1}{n}\sum_{j=1}^{n} x(t_j)$, where $t_i = a + i\Delta$, then

$$\frac{(y^{(m)}, y^{(m)})}{\Delta^{2m}} \cdot \Delta \geq \frac{4^m \sin^{2m}\left(\frac{\pi}{n}\right)}{\Delta^{2m}} \cdot (y, y)\Delta.$$

The result now follows when we let $n \to \infty$.

A careful inspection of the proof of Theorem 1 will show that we only used the periodic extension property of the n-vector x on $x_{1-\left[\frac{m}{2}\right]}, x_{2-\left[\frac{m}{2}\right]}, \ldots, x_0$ and $x_{n+1}, x_{n+2}, \ldots, x_{n+m-\left[\frac{m}{2}\right]}$. Hence we may weaken our requirement $x(t)$ has period $b-a$ to a condition on the end points, namely $x(a) = x(b)$, $x'(a) = x'(b)$, $\ldots$, $x^{(m-1)}(a) = x^{(m-1)}(b)$.

Halperin and Pitt [3], Müller [5], Nirenberg [6], and Redheffer [7] have developed inequalities of the form:

$$\int_a^b \{x'(t)\}^2 dt \leq \alpha \int_a^b \{x''(t)\}^2 dt + H(\alpha) \int_a^b \{x(t)\}^2 dt,$$

where $x(t) \in C^2[a, b]$, $\alpha > 0$. Redheffer obtained the best possible value of $H(\alpha)$. We will now develop generalizations of the form:

$$\int_a^b \{x^{(k)}(t)\}^2 dt \leq \alpha \int_a^b \{x^{(m)}(t)\}^2 dt + H_{k,m}(\alpha) \int_a^b \{x(t)\}^2 dt,$$

$1 \leq k < m$ under the further restriction that $x(a) = x(b)$, $x'(a) = x'(b)$, $\ldots$, $x^{(m-1)}(a) = x^{(m-1)}(b)$. These inequalities will be established by taking limits of the appropriate discrete inequalities.

Incidentally we observe that $H_{1,2}(\alpha)$ for our periodic case is much smaller than Redheffer's general value of $\frac{1}{\alpha}+\frac{12}{(b-a)^2}$.

If we let $P(t)=\theta^k t^k-\alpha\theta^m t^m$, then we wish to maximize $P(t)$ on $[\lambda_n, \lambda_1]=\left[0, 4\sin^2\left(\left[\frac{n}{2}\right]\frac{\pi}{n}\right)\right]$. Since $P'(t)=k\theta^k t^{k-1}-m\alpha\theta^m t^{m-1}$, if we assume that $\theta>0$ then the maximum value of $P(t)$ on $t\geq 0$ occurs at $t=\frac{1}{\theta}\left(\frac{k}{m\alpha}\right)^{1/m-k}$ Thus $\sup \lambda_n \leq t \leq \lambda_1\{P(t)\} \leq P\left(\frac{1}{\theta}\left\{\frac{k}{m\alpha}\right\}^{1/m-k}\right)=\left(\frac{k}{m\alpha}\right)^{k/m-k}\left(1-\frac{k}{m}\right)$.

An inspection of the graph of $P(t)$ will yield the following cases:

If $\lambda_1 \geq \frac{1}{\theta}\left(\frac{k}{m\alpha}\right)^{1/m-k}$ then

$$\sup_{\lambda_n \leq t} \leq \lambda_1\{P(t)\}=P\left(\frac{1}{\theta}\left[\frac{k}{m\alpha}\right]^{1/m-k}\right)=\left(\frac{k}{m\alpha}\right)^{k/m-k}\left(1-\frac{k}{m}\right).$$

If $\lambda_1 \leq \frac{1}{\theta}\left(\frac{k}{m\alpha}\right)^{1/m-k}$ then

$$\sup \lambda_n \leq t \leq \lambda_1\{P(t)\}=P(\lambda_1)=P\left(4\sin^2\left(\left[\frac{n}{2}\right]\frac{\pi}{n}\right)\right)$$

$$=4^k\theta^k\sin^{2k}\left(\left[\frac{n}{2}\right]\frac{\pi}{n}\right)\left\{1-4^{m-k}\theta^{m-k}\alpha\sin^{2m-2k}\left(\left[\frac{n}{2}\right]\frac{\pi}{n}\right)\right\}.$$

If n is even then $P(\lambda_1)=4^k\theta^k\{1-4^{m-k}\theta^{m-k}\alpha\}$.

Hence we have the following result.

THEOREM 2. *If* x *is a periodically extended* n*-vector,* α *and* θ *are positive constants,* $1 \leq k < m$ *then we have the following inequality:*

$$\theta^k(x^{(k)}, x^{(k)}) \leq \alpha\theta^m(x^{(m)}, x^{(m)})+\left(\frac{k}{m\alpha}\right)^{k/m-k}(1-k/m)(x, x).$$

If $4 \leq \frac{1}{\theta}\left(\frac{k}{m\alpha}\right)^{1/m-k}$, *then we may improve the above inequality to:*

$$\theta^k(x^{(k)}, x^{(k)})-\alpha\theta^m(x^{(m)}, x^{(m)}) \leq 4^k\theta^k\{1-4^{m-k}\theta^{m-k}\alpha\}\cdot(x, x).$$

PROOF:

We note that

$$4 \leq \frac{1}{\theta}\left(\frac{k}{m\alpha}\right)^{1/m-k}$$

certainly implies that

$$4\sin^2\left(\left[\frac{n}{2}\right]\frac{\pi}{n}\right) \leq \frac{1}{\theta}\left(\frac{k}{m\alpha}\right)^{1/m-k}$$

and $P(t)$ is increasing on the interval $\left[0, \frac{1}{\theta}\left(\frac{k}{m\alpha}\right)^{1/m-k}\right]$.

If we take $\theta=\left(\frac{n+1}{b-a}\right)^2$ and let $n\to\infty$ then we obtain a continuous analog of Theorem 2.

COROLLARY 2.1. *If* $x(t)\in C^m[a, b]$, $1\leq k<m$, $x(a)=x(b)$, $x'(a)=x'(b)$, . . ., $x^{(m-1)}(a)=x^{(m-1)}(b)$, *and if* $\alpha>0$ *then we have the following inequality:*

$$\int_a^b \{x^{(k)}(t)\}^2dt\leq\alpha\int_a^b \{x^{(m)}(t)\}^2dt+\left(\frac{k}{m\alpha}\right)^{k/m-k}(1-k/m)\int_a^b \{x(t)\}^2dt.$$

If we let $k=1$, $m=2$ *then our constant has the value* $\frac{1}{4\alpha}$ *as compared to the Redheffer value of* $\frac{1}{\alpha}+\frac{12}{(b-a)^2}$.

One may now inquire about the possibility of equality in Corollary 2.1; by our previous discussion we see that this is possible if and only if

$$\lim_{n\to\infty}[1\leq\max_i\leq n\{\theta^k\lambda_i^k-\alpha\theta^m\lambda_i^m\}]=\left(\frac{k}{m\alpha}\right)^{k/m-k}\left(1-\frac{k}{m}\right).$$

However a brief investigation shows that the above expression is not always satisfied, hence Corollary 2.1 is in general not best possible. A little more work will yield the best inequality.

We notice that the smaller eigenvalues of P approach zero as $1/n^2$ and the positive root of $\theta^k t^k-\alpha\theta^m t^m$ exhibits a similar behavior:

$$\lambda_n=0,\ \lambda_{n-1}=4\sin^2\left(\frac{\pi}{n}\right)\approx\frac{4\pi^2}{n^2},\ \lambda_{n-3}=4\sin^2\left(\frac{2\pi}{n}\right)\approx\frac{8\pi^2}{n^2},\ \ldots$$

$$\text{root}=\frac{(b-a)^2}{(n+1)^2}\cdot\left(\frac{1}{\alpha}\right)^{1/m-k}.$$

Hence if we set

$$L=\left[\frac{b-a}{2\pi}\cdot\left(\frac{1}{\sqrt{\alpha}}\right)^{1/m-k}\right],$$

where [ ] is the greatest integer function, then L is the limiting number of distinct eigenvalues of P which are greater than 0 and less than or equal to the root.

If $L=0$, which occurs when $\alpha>\left(\frac{b-a}{2\pi}\right)^{2m-2k}$, then

$$\lim_{n\to\infty}[1\leq\max_i\leq n\{\theta^k\lambda_i^k-\alpha\theta^m\lambda_i^m\}]=0,$$

yielding the best possible result:

COROLLARY 2.2. *Let* $x(t)$, k, m *be as in Corollary 2.1 and let* $\alpha>\left(\frac{b-a}{2\pi}\right)^{2m-2k}$, *then*

$$\int_a^b \{x^{(k)}(t)\}^2dt\leq\alpha\int_a^b \{x^{(m)}(t)\}^2dt.$$

If we add the auxiliary condition $\int_a^b x(t)dt = 0$, then

$$\lim_{n\to\infty}\left[\max_{1\le i\le n-1}\{\theta^k\lambda_i^k - \alpha\theta^m\lambda_i^m\}\right] = \left(\frac{2\pi}{b-a}\right)^{2k} - \alpha\left(\frac{2\pi}{b-a}\right)^{2m} \le 0$$

and we obtain the best possible result:

COROLLARY 2.3. *Let* x(t), k, m, α *be as in Corollary 2.2 and let* x *also satisfy* $\int_a^b x(t)dt = 0$, *then*

$$\int_a^b \{x^{(k)}(t)\}^2 dt \le \alpha\int_a^b \{x^{(m)}(t)\}^2 dt + \left\{\left(\frac{2\pi}{b-a}\right)^{2k} - \alpha\left(\frac{2\pi}{b-a}\right)^{2m}\right\}\cdot\int_a^b \{x(t)\}^2 dt.$$

If $L \ge 1$ then there are limiting eigenvalues between zero and the positive root. Hence if we set

$$J = \left[\frac{b-a}{2\pi}\cdot\left(\sqrt{\frac{k}{m\alpha}}\right)^{1/m-k}\right],$$

then J is the limiting number of distinct eigenvalues of P which are greater than 0 and less than or equal to the maximum of $\theta^k t^k - \alpha\theta^m t^m$.

Thus if

$$\left[\frac{b-a}{2\pi}\cdot\left(\sqrt{\frac{k}{m\alpha}}\right)^{1/m-k}\right] = \frac{b-a}{2\pi}\cdot\left(\sqrt{\frac{k}{m\alpha}}\right)^{1/m-k}$$

then

$$\lim_{n\to\infty}\left[\max_{1\le i\le n}\{\theta^k\lambda_i^k - \alpha\theta^m\lambda_i^m\}\right] = \left(\frac{k}{m\alpha}\right)^{k/m-k}(1-k/m)$$

and Corollary 2.1 is best possible. Otherwise we may replace

$$\left(\frac{k}{m\alpha}\right)^{k/m-k}(1-k/m) \text{ by } \max\left\{\left(\frac{2J\pi}{b-a}\right)^{2k} - \alpha\left(\frac{2J\pi}{b-a}\right)^{2m}, \left(\frac{2(J+1)\pi}{b-a}\right)^{2k} - \alpha\left(\frac{2(J+1)\pi}{b-a}\right)^{2m}\right\}.$$

We may summarize the above results by the following best possible inequalities.

COROLLARY 2.4. *If* $x(t)\in C^m[a, b], 1 \le k < m$, $x(a) = x(b)$, $x'(a) = x'(b)$, . . ., $x^{(m-1)}(a) = x^{(m-1)}(b)$, *and if* $\alpha > 0$ *then*

$$\int_a^b \{x^{(k)}(t)\}^2 dt \le \alpha\int_a^b \{x^{(m)}(t)\}^2 dt + H_{k,m}(\alpha)\int_a^b \{x(t)\}^2 dt,$$

where

$$H_{k,m}(\alpha) = 0 \text{ for } \alpha > \left(\frac{b-a}{2\pi}\right)^{2m-2k},$$

$$H_{k,m}(\alpha) = \left(\frac{k}{m\alpha}\right)^{k/m-k}(1-k/m) \text{ if } \frac{b-a}{2\pi}\cdot\left(\sqrt{\frac{k}{m\alpha}}\right)^{1/m-k}$$

is a positive integer, and otherwise

$$H_{k,m}(\alpha)=\max\left\{\left(\frac{2J\pi}{b-a}\right)^{2k}-\alpha\left(\frac{2J\pi}{b-a}\right)^{2m},\left(\frac{2(J+1)\pi}{b-a}\right)^{2k}-\alpha\left(\frac{2(J+1)\pi}{b-a}\right)^{2m}\right\}.$$

COROLLARY 2.5. *Let* x(t), k, m, α *be as in Corollary* 2.4 *and let* x *also satisfy* $\int_a^b x(t)dt=0$, *then*

$$\int_a^b\{x^{(k)}(t)\}^2dt\leq\alpha\int_a^b\{x^{(m)}(t)\}^2dt+G_{k,m}(\alpha)\int_a^b\{x(t)\}^2dt,$$

where

$$G_{k,m}(\alpha)=\left(\frac{2\pi}{b-a}\right)^{2k}-\alpha\left(\frac{2\pi}{b-a}\right)^{2m}\text{ for }\alpha>\left(\frac{b-a}{2\pi}\right)^{2m-2k},$$

and otherwise

$$G_{k,m}(\alpha)=H_{k,m}(\alpha).$$

It is not difficult to see that $H_{k,m}(\alpha)$ and $G_{k,m}(\alpha)$ are piecewise linear functions of α, which are monotonic decreasing, and that the jumps in the derivatives of $H_{k,m}$ and $G_{k,m}$ occur for those positive numbers α_i which yield

$$\left(\frac{2\pi i}{b-a}\right)^{2k}-\alpha_i\left(\frac{2\pi i}{b-a}\right)^{2m}=\left(\frac{2\pi i+2\pi}{b-a}\right)^{2k}-\alpha_i\left(\frac{2\pi i+2\pi}{b-a}\right)^{2m},$$

where i is a nonnegative integer. We have $\infty>\alpha_0>\alpha_1>\ldots>0$ and $\lim_{i\to\infty}\alpha_i=0$. Note that

$$\int_a^b\left\{\sin^{(k)}\left(\frac{2\pi it}{b-a}\right)\right\}^2dt=\alpha\int_a^b\left\{\sin^{(m)}\left(\frac{2\pi it}{b-a}\right)\right\}^2dt+H_{k,m}(\alpha)\int_a^b\left\{\sin\left(\frac{2\pi it}{b-a}\right)\right\}^2dt,$$

for $\alpha_i\leq\alpha\leq\alpha_{i-1}$. This accounts for the piecewise linearity of $H_{k,m}$ and $G_{k,m}$ and will be a great importance in section 4.

At this stage it is apparent that there is an unlimited number of possible inequalities, in fact every polynomial will give discrete inequalities and polynomials in θt yield continuous analogs.

It is known, see for example [1], that if $f(t)$ is represented by a power series with real coefficients and if A is a real symmetric matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ which lie strictly within the circle of convergence of the power series for $f(t)$, then $f(A)$ is real symmetric and has eigenvalues $f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n)$. This result enables us to extend our polynomial inequalities to discrete inequalities on analytic functions.

THEOREM 3. *If* x *is a periodically extended* n*-vector and if* $f(t)=\sum_{m=0}^{\infty}a_mt^m$, *where the* a_m *are real numbers and the series converges on a set containing* [0, 4] *in its interior—if* n *is odd then we need only require convergence on a set containing* $\left[0, 4\sin^2\left(\left[\frac{n}{2}\right]\frac{\pi}{n}\right)\right]$ *in its interior—then*

$$\inf_{[0\leq t\leq 4\{f(t)\}]}(x,x)\leq\sum_{m=0}^{\infty}a_m(x^{(m)},x^{(m)})\leq\sup_{[0\leq t\leq 4\{f(t)\}]}(x,x).$$

The following inequalities are examples of the application of Theorem 3.

COROLLARY 3.1. *If* x *is a periodically extended* n*-vector then*

$$\sum_{m=1}^{\infty} \frac{1}{m!} (x^{(m)}, x^{(m)}) \leq (e^4 - 1)(x, x).$$

COROLLARY 3.2. *If* x *is a periodically extended* n*-vector and* $\theta \geq 0$ *then*

$$-(x, x) \leq \sum_{m=0}^{\infty} (-1)^m \frac{\theta^{2m+1}}{(2m+1)!} (x^{(2m+1)}, x^{(2m+1)}) \leq (x, x).$$

PROOF:

$$\sin t = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} t^{2m+1} \text{ and } |\sin t| \leq 1.$$

COROLLARY 3.3. *If* $x(t) \in C^{\infty}[b-a]$ *and* $x(t)$ *has period* $b-a$ *then*

$$-\int_a^b \{x(t)\}^2 dt \leq \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \int_a^b \{x^{(2m+1)}(t)\}^2 dt \leq \int_a^b \{x(t)\}^2 dt.$$

4. Inequalities of Kolmogoroff Type

In analysis one frequently wants to obtain inclusion regions for the norms of functions; i.e., if M is in the inclusion region, then there exists a function f, from a certain set of functions, with norm of f equal to M. Kolmogoroff [4] established a result of this type using the uniform norm over the function set $C^m[0, \infty]$. If we set $M_k(x) = \sup_{0 \leq t \leq \infty} \{|x^{(k)}(t)|\}$ $(k = 0, 1, 2, \ldots, m)$, then Kolmogoroff [4] gave a necessary and sufficient condition "in order that to a triple of positive numbers M_0, M_k, $M_m (0 < k < m)$ there should correspond a function $x(t)$ for which

$$M_0 = M_0(x),\ M_k = M_k(x),\ M_m = M_m(x)."$$

In this section we will establish a similar result on a triple of positive numbers relative to the square norm, where our set of functions will be $C^m[a, b] \cap \{x | x(a) = x(b),\ x'(a) = x'(b), \ldots, x^{(m-1)}(a) = x^{(m-1)}(b)\}$. We will also add the restriction $\int_a^b x(t)dt = 0$ in order to eliminate the constant function.

If $x(t)$ is an element of our function class, then we set

$$A_k(x) = \int_a^b \{x^{(k)}(t)\}^2 dt \qquad (k = 0, 1, 2, \ldots, m).$$

We may assume without loss of generality that we have normalized x so that $A_0(x) = 1$. We will establish the following result:

THEOREM 4. *In order that to a pair of numbers* a_k, a_m *there should correspond a function* $x(t) \in C^m[a, b] \cap \{x | x(a) = x(b), x'(a) = x'(b), \ldots, x^{(m-1)}(a) = x^{(m-1)}(b), \int_a^b x(t)dt = 0\}$ *for which*

$$A_0(x) = 1,\ A_k(x) = a_k,\ A_m(x) = a_m$$

it is necessary and sufficient that

$$a_k \geq \left(\frac{2\pi}{b-a}\right)^{2k},\ a_m \geq \left(\frac{2\pi}{b-a}\right)^{2m},\ \textit{and}$$

$$a_k - \alpha a_m \leq G_{k,m}(\alpha) \text{ for all } \alpha > 0.$$

PROOF of THEOREM 4:

The necessity of the conditions is a direct consequence of Corollaries 1.1 and 2.5.

The proof of the sufficiency will be divided into several cases and we will use the notation of section 3 throughout. We will denote the normalized $\sin\left(\frac{2\pi i t}{b-a}\right)$ by $S_i(t)$. We recall that on the interval $\alpha_i \leq \alpha \leq \alpha_{i-1}$ we have $A_k(S_i) - \alpha A_m(S_i) = G_{k,m}(\alpha)$.

CASE 1: $a_k - \alpha a_m = G_{k,m}(\alpha)$ for at least two values of α, say $\alpha = t_1$ and $\alpha = t_2$.

We immediately have that $a_k - \alpha a_m = G_{k,m}(\alpha)$ for all $\alpha \in [t_1, t_2]$. Since we know that the S_i are maximizing functions there is an integer i with $\alpha_i \leq t_1 < t_2 \leq \alpha_{i-1}$ and $a_k - \alpha a_m = G_{k,m}(\alpha) = A_k(S_i) - \alpha A_m(S_i)$ for $\alpha_i \leq \alpha \leq \alpha_{i-1}$. Thus $a_k = A_k(S_i)$, $a_m = A_m(S_i)$.

CASE 2: $a_k - \alpha a_m = G_{k,m}(\alpha)$ for only one value of α, say $\alpha = t_1$.

We immediately have that $t_1 = \alpha_i$ for exactly one integer i.
Thus

$$a_k - \alpha_i a_m = G_{k,m}(\alpha_i) = A_k(S_i) - \alpha_i A_m(S_i) = A_k(S_{i+1}) - \alpha_i A_m(S_{i+1}).$$

Also since $a_k - \alpha a_m \leq G_{k,m}(\alpha)$ for all $\alpha > 0$ we must have $A_m(S_i) < a_m < A_m(S_{i+1})$. Thus there exist b_i, b_{i+1} with $0 < b_i,\ b_{i+1} < 1$, $b_i^2 + b_{i+1}^2 = 1$, and $a_m = A_m(b_i S_i + b_{i+1} S_{i+1}) = b_i^2 A_m(S_i) + b_{i+1}^2 A_m(S_{i+1})$. [Note that the last equality follows from the orthogonality of S_i and all its derivatives to S_{i+1} and all its derivatives.] Now,

$$A_k(b_i S_i + b_{i+1} S_{i+1}) - \alpha_i a_m = A_k(b_i S_i + b_{i+1} S_{i+1}) - \alpha_i A_m(b_i S_i + b_{i+1} S_{i+1})$$

$$= b_i^2 [A_k(S_i) - \alpha_i A_m(S_i)] + b_{i+1}^2 [A_k(S_{i+1}) - \alpha_i A_m(S_{i+1})] = (b_i^2 + b_{i+1}^2)[a_k - \alpha_i a_m].$$

Hence $a_k = A_k(b_i S_i + b_{i+1} S_{i+1})$.

CASE 3: $a_k - \alpha a_m < G_{k,m}(\alpha)$ for all $\alpha > 0$ and $a_m \geq \left(\frac{4\pi}{b-a}\right)^{2m} = A_m(S_2)$.

We know that there is exactly one integer $i \geq 2$ with $\left(\frac{2i\pi}{b-a}\right)^{2m} \leq a_m < \left(\frac{2(i+2)\pi}{b-a}\right)^{2m}$ or $A_m(S_i) \leq a_m < A_m(S_{i+1})$. If we repeat the construction of Case 2 we can find b_i and b_{i+1} with $a_m = A_m(b_i S_i + b_{i+1} S_{i+1})$ and since $a_k - \alpha_i a_m < G_{k,m}(\alpha_i) = A_k(b_i S_i + b_{i+1} S_{i+1}) - \alpha_i A_m(b_i S_i + b_{i+1} S_{i+1}) = A_k(b_i S_i + b_{i+1} S_{i+1}) - \alpha_i a_m$ we have

$$a_k < A_k(b_i S_i + b_{i+1} S_{i+1}).$$

By our conditions on a_k and a_m there is an integer $j > i+1$ and an $\bar{\alpha} > 0$ such that $A_k(S_j) - \bar{\alpha} A_m(S_j) = A_k(S_1) - \bar{\alpha} A_m(S_1) < a_k - \bar{\alpha} a_m$. Now $A_m(S_1) < A_m(S_i) \leq a_m < A_m(S_{i+1}) < A_m(S_j)$, hence there are b_1, b_j with $a_m = A_m(b_1 S_1 + b_j S_j)$ and since

$$a_k - \bar{\alpha} a_m > G_{k,m}(\bar{\alpha}) = A_k(b_1 S_1 + b_j S_j) - \bar{\alpha} A_m(b_1 S_1 + b_j S_j) = A_k(b_1 S_1 + b_j S_j) - \bar{\alpha} a_m$$

we have

$$a_k > A_k(b_1S_1 + b_jS_j).$$

By the orthogonality of $b_1S_1 + b_jS_j$ and $b_iS_i + b_{i+1}S_{i+1}$ there are constants c and d with $c^2 + d^2 = 1$ for normalization and

$$a_m = A_m(cb_1S_1 + CB_jS_j + db_iS_i + db_{i+1}S_{i+1}),$$

$$a_k = A_k(cb_1S_1 + cb_jS_j + db_iS_i + db_{i+1}S_{i+1}).$$

CASE 4: $a_k - \alpha a_m < G_{k,m}(\alpha)$ for all α and $a_m < \left(\frac{4\pi}{b-a}\right)^{2m} = A_m(S_2)$.

If we repeat the process of Cases 2 and 3 we can find b_1, b_2 such that $a_m = A_m(b_1S_1 + b_2S_2)$ and $a_k < A_k(b_1S_1 + b_2S_2)$.

We cannot simply repeat the process of Case 3 because $b_1S_1 + b_2S_2$ will not be orthogonal to $b_1'S_1 + b_2'S_2$. However we note that the normalized multiple of $\cos\left(\frac{2\pi t}{b-a}\right)$, which we denote by $C_1(t)$, also satisfies $A_k(C_1) - \alpha A_m(C_1)$ for $\alpha_1 \leq \alpha \leq \alpha_0$ and is orthogonal to all the S_i. Hence we can find c_1, c_j such that

$$a_m = A_m(c_1C_1 + c_jS_j) \text{ and } a_k > A_k(c_1C_1 + c_jS_j)$$

and we continue as in Case 3.

Thus we have proven Theorem 4.

5. References

[1] S. Afriat, Analytic functions of matrices, Proc. Camb. Phil. Soc. **55,** 51–56 (1956).

[2] K. Fan, O. Taussky, J. Todd, Discrete analogs of inequalities of Wirtinger, Monatshefte für Math. **59,** 73–90 (1955).

[3] I. Halperin and H. Pitt, Integral inequalities connected with differential operators, Duke Math. J. **4,** 613–625 (1938).

[4] A. Kolmogoroff, On inequalities between the upper bounds of the successive derivatives of an arbitrary function on an infinite interval, Amer. Math. Soc. Translation No. 4, 1949.

[5] W. Müller, Über eine Ungleichung zwischen den Normen von f, f' und f^n, Math. Zeitschr. **78,** 420–422 (1962).

[6] L. Nirenberg, Remarks on strongly elliptic partial differential equations, Appendix, Comm. Pure and Appl. Math. **8,** 649–675 (1955).

[7] R. Redheffer, Über eine beste Ungleichung zwischen den Normen von f, f', f'', Math. Zeitschr. **80,** 390–397 (1963).

[8] D. E. Rutherford, Some continuant determinants arising in physics and chemistry, I., Proc. Royal Soc. Edinburgh, **52A,** 229–236 (1947).

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Publications of the National Bureau of Standards*

Selected Abstracts

A note on the numerical evaluation of thermal radiation characteristics of diffuse cylindrical and conical cavities, B. A. Peavy, *J. Res. NBS* **70C** *(Engr. and Instr.) No. 2, 139–147 (Apr.–June 1966).*

Presented are methods that avoid the need to employ an extrapolation technique in the region of the critical points for evaluation of the apparent emissivity of diffuse cylindrical and conical cavities. The methods involve appropriate substitutions in the integrands of integral equations that are used in analytical solutions for determining the thermal radiation characteristics of diffuse and conical cavities. Equations for either isothermal or nonisothermal surface temperature conditions are provided in a direct form for computations. Numerical results are presented for a general linear temperature distribution along the length of a cylindrical cavity. The method is equally applicable for the solution of other problems in integral equations where discontinuities are encountered.

Two-dimensional probability distribution in a turbulent field, F. N. Frenkiel and P. S. Klebanoff, *Phys. Fluids* **8,** *No. 12, 2291–2293 (Dec. 1965).*

Using high speed computing techniques measurements were made for the first time of a two-dimensional probability distribution in a turbulent field generated by a grid in a wind tunnel. The two-dimensional probability distribution of simultaneous turbulent velocities at two points separated transversely to the mean flow (in the initial stage of decay) is presented, and the departure from the customarily assumed Gaussian distribution is shown.

Propagation of electromagnetic pulses in terrestrial waveguides, J. R. Wait, *IEEE Trans. Ant. Prop.* **AP–13,** *No. 6, 904–918 (Nov. 1965).*

The propagation of electromagnetic waves over the earth's surface is considered under transient conditions. The source is taken to be a vertical electric dipole whose current moment is suddenly established. The build-up of the radiated field is calculated under various assumed conditions. It is shown, even in the absence of an ionospherically reflected wave, that the influence of earth curvature has a pronounced effect on the distortion of the original pulse shape. For great distances (i.e., $d > 2000$ km), it is found to be more convenient to regard the field as a sum of modes. Particular attention is given to the transient characteristics of the dominant mode as a function of the source waveform.

The analysis, synthesis, and description of biological images, L. E. Lipkin, W. C. Watt, and R. A. Kirsch, *Ann. N.Y. Acad. Sci.* **128,** *No. 3, 984–1012 (Jan. 31, 1966).*

There may be distinguished two approaches to characterizing the properties of biological images: the statistical approach traditionally used for mechanized image-processing, and the articular approach generally used among biologists. We discuss both approaches, but focus attention on the second, for we hold that an articular approach allows the expression of much that is effectively impossible to express in the form of numerical measurements; and we know of no inherent reason why a computer cannot deal with information of a nonquantitative nature.

The articulation of biological images necessarily takes place on two fronts: the imposition of an articular structure on the image itself; and the expression of this construal in English sentences. We will study both kinds of articulation, and we will suggest how they may be brought together, in the form of linked pictorial and linguistic *grammars*. The computer system which this paper envisions will be able to analyze a presented image with respect to a pictorial grammar, and to formulate and accept descriptions of that image, in English sentences, with respect to a linguistic grammar. It will be able to present pictorial instances of English descriptions, and in other ways to respond to English directives.

Status of linear relations among heats of transport, R. E. Howard and A. B. Lidiard, *J. Chem. Phys.* **43,** *No. 11, 4158–4160 (Dec. 1, 1965).*

It is shown that in situations where the fluxes of the various components of a system are not linearly independent, previously derived linear relations among the heats of transport contain no physical information, but serve only to complete the definition of these quantities. Other defining equations are possible but physical predictions are unaffected by the choice so presented. This resolves the paradox of the apparent inconsistency of these linear relations with kinetic theories of heats of transport.

Other NBS Publications

J. Res. NBS 70A, (Phys. and Chem.), No. 3, (May–June 1966), $1.00.

Thermal relaxation of Brillouin scattering in liquids. R. D. Mountain.
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Ellipsoidal mirror reflectometer. S. Thomas Dunn, J. C. Richmond, and J. A. Wiebelt.
Reference tables for the Pt-30 percent Rh versus Pt-6 percent Rh thermocouple. G. W. Burns and J. S. Gallagher.
Coaxial power meter calibration using a waveguide standard. G. F. Engen.
A note on the numerical evaluation of thermal radiation characteristics of diffuse cylindrical and conical cavities. B. A. Peavy. (See above abstract.)

J. Res. NBS 70C, (Engr. and Instr.), No. 3, (July–Sept. 1966), $1.00.

Some harmonic properties of an oscillating Fabry-Perot interferometer. M. Gadsden and H. M. Williams.
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Magnetic transformation and the influence of plastic strain on the shear modulus of Fe-Cr-Ni alloys. R. P. Mikesell and R. P. Reed.

NBS standard frequency and time services, NBS Misc. Publ. 236, 1966 edition (1966), 15 cents (supersedes previous editions).

Statistical association methods for mechanized documentation. Symposium Proceedings Washington 1964, M. E. Stevens, V. E. Giuliano, and L. B. Heilprin, NBS Misc. Publ. 269 (Dec. 15, 1965), $2.75.

Report of the 50th National Conference on Weights and Measures 1965, NBS Misc. Publ. 272 (Apr. 1, 1966), $1.00.

Photonuclear data index, NBS Misc. Publ. 277 (Apr. 1, 1966), 55 cents.

Bibliography on atomic transition probabilities, B. W. Glennon and W. L. Wiese, NBS Misc. Publ. 278 (Apr. 22, 1966), 55 cents, (supersedes Mono. 50).

Radio meteorology, B. R. Bean and E. J. Dutton, NBS Mono. 92, (March 1, 1966), $2.75.

OMNITAB. A computer program for statistical and numerical analysis, J. Hilsenrath, G. G. Ziegler, C. G. Messina, P. J. Walsh, and R. J. Herbold, NBS Handb. 101 (March 4, 1966), $3.00.

Copper wire tables, NBS Handb. 100 (Feb. 21, 1966), 50 cents, supersedes Circ. 31.

List of errata to accompany the first, second, and third printings of NBS Applied Mathematics Series 55, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, ed. M. Abramowitz and I. A. Stegun (April, 1966). This errata is free to holders of AMS55, upon request from the Publications Section, A617 Administration Building, National Bureau of Standards, Washington, D.C. 20234.

Doors as barriers to fire and smoke, H. Shoub and D. Gross, Bldg. Sci. Series **3,** (Mar. 25, 1966), 15 cents.

Quarterly radio noise data September, October, November, 1964, W. Q. Crichlow, R. T. Disney, and M. A. Jenkins, NBS Tech. Note 18-24 (Feb. 10, 1966), 50 cents. (Formerly the Central Radio Propagation Laboratory of the National Bureau of Standards, now Environmental Science Services Administration, Boulder, Colo.)

Procedures for precise determination of thermal radiation properties, November 1963 to October 1964, J. C. Richmond, S. T. Dunn, D. P. DeWitt, and W. D. Hayes, Jr., NBS Tech. Note 267, (Dec. 17, 1965), 45 cents.

Earth's field static calibrator for accelerometers, P. S. Lederer and J. S. Hilten, NBS Tech. Note 269 (Feb. 1, 1966), 20 cents.

Nuclear and radiation standards of importance to the National Atomic Energy Program, H. W. Koch, J. H. Donnert, W. W. Havens, Jr., G. L. Rogosa, and L. Rosen, NBS Tech. Note 283 (Mar. 31, 1966), 35 cents.

An outline for cooperative action on the determination of x-ray mass attenuation coefficients in the wavelength range from 0.5 to 100 Å (25 to 0.12 keV), K. F. J. Heinrich, NBS Tech. Note 284 (Feb. 21, 1966), 15 cents.

Computation of the permeability and permittivity of a relatively small ring sample in a toroidal coil, E. G. Johnson, Jr., NBS Tech. Note 311 (Jan. 25, 1966), 30 cents.

Disclosures on magnetic tape handler focus detector, multi-pen recorder, signal averaging filter, electrostatic RMS voltmeter, back-up ring for O-rings, and low temperature solid state thermometer, ed. D. Robbins and A. J. Englert, NBS Tech. Note 287 (Apr. 8, 1966), 25 cents.

An atlas of solar flare effects in the ionosphere observed with a high-frequency Doppler technique, September 1960–December 1962, D. M. Baker, NBS Tech. Note 326 (Dec. 1, 1965), 60 cents. (Formerly the Central Radio Propagation Laboratory of the National Bureau of Standards, now Environmental Science Services Administration, Boulder, Colo.)

The seasonal variation of nightglow NaI 5890–96 A, [01] 5577 A and [01] 6300 A in the tropics, L. L. Smith and R. W. Owen, NBS Tech. Note 329 (Jan. 10, 1966), 35 cents. (Formerly the Central Radio Propagation Laboratory of the National Bureau of Standards, now Environmental Science Services Administration, Boulder, Colo.)

Computations of the antenna cut-back factor for LF radio waves, D. C. Hyovalti, NBS Tech. Note 330 (Nov. 26, 1965), 30 cents. (Formerly Central Radio Propagation Laboratory of the National Bureau of Standards, now Environmental Science Services Administration, Boulder, Colo.)

A 100 KW 2–25 Mc/s distributed amplifier, designed for use with 10 kW ionospheric sounders, W. B. Harding, M. W. Woodward, and J. C. Carroll, NBS Tech. Note 331 (Feb. 15, 1966), 20 cents. (Formerly Central Radio Propagation Laboratory of the National Bureau of Standards, now Environmental Science Services Administration, Boulder, Colo.)

A review of zinc oxide-eugenol type filling materials and cements, G. M. Brauer, Rev. Belge. Med. Dent.-Belg. Tijds. Tandheelk **20,** No. 3, 323–364 (1965).

Application of a high-intensity, multi-slit Raleigh interferometer to sedimentation studies, I. H. Billick and R. J. Bowen, J. Phys. Chem. **69,** 4024–4030 (1965).

Crystal and molecular structure of $C_4B_{20}H_{22}$, bis (*o*-dodecacarborane), L. H. Hall, A. Perloff, F. A. Mauer, and S. Block, J. Chem. Phys. **43,** No. 11, 3911–3917 (Dec. 1965).

Dissociation of ammonium ion in methanol-water solvents, M. Paabo, R. G. Bates, and R. A. Robinson, J. Phys. Chem. **70,** No. 1, 247–251 (1966).

Electrochemistry, W. J. Hamer, Book, Encyclopedia of Physics, pp. 200–203 (Reinhold Publ. Corp., New York, N.Y., 1966).

Kinetics of the hydroxyl radical in aqueous solution, F. Sicilio, R. E. Florin, and L. A. Wall, J. Phys. Chem. **70,** No. 47, 47–52 (Jan. 1966).

Manganese-hydrogen bond distance in $HMn(CO)_5$, T. C. Farrar, W. Ryan, A. Davison, and J. W. Faller, J. Am. Chem. Soc. **88,** 184 (Jan. 5, 1966).

Matrix-isolation infrared spectrum of the free radical CCO, M. E. Jacox, D. E. Milligan, N. G. Moll, and W. E. Thompson, J. Chem. Phys. **43,** No. 10, 3734–3746 (Nov. 1965).

Matrix-isolation infrared spectrum of the free radical NH_2, D. E. Milligan and M. E. Jacox, J. Chem. Phys. **43,** No. 12, 3487–4493 (Dec. 15, 1965).

Matrix isolation study of the photolysis of cyanogen azide. The infrared and ultraviolet spectra of the free radical NCN, D. E. Milligan, M. E. Jacox, and A. Bass, J. Chem. Phys. **43,** No. 9, 3149–3160 (Nov. 1, 1965).

Patterns of thinking in searching patent applications by manual and machine-assisted methods, J. R. Cornog and P. P. Ellis, J. Chem. Doc. **5,** 215–225 (Nov. 1965).

Photolysis of cyclobutane at photo energies below and above the ionization energy, R. D. Doepker and P. Ausloos, J. Chem. Phys. **43,** No. 11, 3814–3819 (Dec. 1, 1965).

Separation of pyrenediones by column chromatography, A. J. Fatiadi, J. Chromatog. **20,** 319–324 (1965).

The effect of gamma-irradiation on a polyamide, V. J. Kransnansky, M. S. Parker, and R. E. Florin, J. Phys. Chem. **70,** No. 1, 40–46 (Jan. 1966).

The thermal degradation mechanism of polystryene, L. A. Wall, S. Straus, J. H. Flynn, D. McIntyre, and R. Simha, J. Phys. Chem. **70,** No. 1, 53–62 (Jan. 1966).

Zone refiner with temperature control, F. Ordway, Anal. Chem. **37,** 1178–1180 (Aug. 1965).

A simple method for making stereoscopic drawings, F. Ordway, Am. Mineralogist **50,** 1496–1498 (1965).

A system for accurate direct and alternating voltage measurements, F. L. Hermach, J. E. Griffin, and E. S. Williams, IEEE Trans. Instr. Meas. **IM-14,** No. 4, 215–224 (Dec. 1965).

Effect of surface roughness on emittance of non-metals, J. C. Richmond, J. Opt. Soc. Am. **56,** No. 2, 253–254, (Feb. 1966).

Comparison of uncertainty ellipses calculated from two spectrophotometric colorimetry methods by an automatic-computer program, I. Nimeroff, J. Opt. Soc. Am. **56,** No. 2, 230–237 (Feb. 1966).

An international comparison of current-ratio standards at audio frequencies, B. L. Dunfee and W. J. M. Moore, IEEE Trans. Instr. Meas. **IM-14,** No. 4, 172–177 (Dec. 1965).

The design and performance of multirange current transformer standards for audio frequencies, B. L. Dunfee, IEEE Trans. Instr. Meas. **IM-14,** No. 4, 190–204 (Dec. 1965).

Color, D. B. Judd, Book, Encyclopedia of Physics, pp. 113–116 (Reinhold Publ. Corp., New York, N.Y., 1966).

Deposition of energy by neutrons in spherical cavities, R. S. Caswell, Radiation Res. **27,** No. 1, 92–107 (Jan. 1966).

High-frequency elastic moduli of simple fluids, R. Zwanzig and R. D. Mountain, J. Chem. Phys. **43,** No. 12, 4464–4471 (Dec. 15, 1965).

Initiation of spherulite growth: The case of concurrent homogeneous and heterogeneous nucleation, F. Gornick, J. Appl. Phys. **36,** No. 10, 3012–3014 (Oct. 1965).

Limits on calorimetric residual entropies of glasses, A. B. Bestul and S. S. Chang, J. Chem. Phys. **43,** No. 12, 4532–4533 (Dec. 15, 1965).

Nuclear moments of Pm^{147}, J. Reader, Phys. Rev. **141,** No. 3, 1123–1128 (Jan. 1966).

Erratum-Nuclear magnetic moment of Pr^{141} from the hyperfine structure of doubly ionized praseodymium, J. Reader and J. Sugar, Phys. Rev. **140,** No. 7AB, AB3 (Dec. 27, 1965).

Photometry, L. E. Barbrow, Book, Encyclopedia of Physics, pp. 516–519 (Reinhold Publ. Corp., New York, N.Y. 1966).

Relativistic electron precipitation into the mesosphere at subauroral latitudes, D. K. Bailey and M. A. Pomerantz, J. Geophys. Res. **70,** No. 23, 5823–5830 (Dec. 1, 1965).

Absolute-velocity mossbuaer spectrometer, J. J. Spijkerman and F. C. Ruegg, Trans. Am. Nucl. Soc. **8,** No. 2, 344 (1966).

Adhesive bonding of various materials to hard tooth tissues. V. The effect of a surface-active comonomer on adhesion to diverse substrates, R. L. Bowen, J. Dental Res. **44,** No. 6, 1369–1373 (Nov.–Dec. 1965).

A simple device for adjusting dental interferometers, H. J. Caul and J. W. Kumpula, J. Dental Res. **44,** No. 6, 1412 (Nov.–Dec. 1965).

Chemical analysis, G. M. Brauer and G. M. Kline, Encyclopedia of Polymer and Technology **3,** 633–665 (John Wiley & Sons, Inc., New York, N.Y., 1965).

Determination of trace amounts of cadmium and silver by substoichiometric radioisotopic dilution analysis, A. R. Landgrebe, L. T. McClendon, and J. R. DeVoe, Trans. Am. Nucl. Soc. **8,** No. 2, 315–316 (1966).

Dissociation of acetic acid-d_3 in aqueous solution and related isotope effects from 0 to 50°, M. Paabo, R. G. Bates, and R. A. Robinson, J. Phys. Chem. **70,** 540–543 (1966).

Magnesium isotope ratios in natural samples, E. J. Catanzaro and T. J. Murphy, J. Geophys. Res. **71,** No. 4, 1271–1274 (Feb. 15, 1966).

NMR studies of inorganic fluorides. I. High-resolution ^{19}F spectra of Si_2F_6 and $(SiF_3)_2O$, R. B. Johannesen, R. C. Farrar, F. E. Brinckman, and T. D. Coyle, J. Chem. Phys. **44,** No. 3, 962–964 (Feb. 1, 1966).

On the limiting shape of the distribution function of lengths of a single polymer molecule with excluded-volume effects, J. Mazur, J. Chem. Phys. **43,** No. 12, 4354–4356 (Dec. 15, 1965).

Viscosity and density of boron trioxide, A. Napolitano, P. B. Macedo, and E. T. Hawkins, J. Am. Ceramic Soc. **48,** No. 12, 613–616 (1965).

Comparison of xenon and carbon arcs as radiation sources for laboratory weathering of asphalts, K. G. Martin, P. G. Campbell, and J. R. Wright, Am. Soc. Testing Mater. Proc. **65,** 809–830 (1965).

Proposed standard for measuring and reporting physical properties of optical materials, G. W. Cleek, F. J. P. Consitt, and W. D. Lawson, Infrared Phys. **5,** 141–160 (Dec. 1965).

Techniques for detection of high-altitude nuclear explosions, A. G. Jean, C. E. Hornback, R. H. Doherty, J. R. Winkelman, and R. T. Moore, Proc. IEEE **53,** No. 12, 2088–2098 (Dec. 1965).

The low temperature removal of small quantities of nitrogen or methane from hydrogen gas by physical adsorption on a synthetic zeolite, A. J. Kidnay and M. J. Hiza, A. I. Ch. E. J. **12,** No. 1, 58–64 (Jan. 1966).

Departures from local thermal equilibrium in a magnetically-driven shock wave, K. L. Eckerle and R. W. P. McWhirter, Phys. Fluids **9,** No. 1, 81–89 (Jan. 1966).

Effect of normal-fluid motion on third sound in liquid-helium films, G. L. Pollack, Phys. Rev. **143,** No. 1, 103–109 (Mar. 4, 1966).

Mossbauer line broadening in SnO_2, R. H. Herber and J. Spijkerman, J. Chem. Phys. **42,** No. 12, 4312–4314 (June 15, 1966).

Multivibrator provides bidirectional output pulses, F. C. Ruegg, Electronics **38,** No. 17, 87 (Aug. 8, 1965).

Precision of cubic lattice parameter measurements by the Kossel technique, H. Yakowitz, Trans. Symp. Electron Probe Analysis, Oct. 12–15, 1964, Wash., D.C., pp. 417–438 (John Wiley & Sons, Inc., New York, N.Y., 1966).

Radiative corrections. I. High-energy bremsstrahlung and pair production, K. Mork and H. Olsen, Phys. Rev. **140,** No. 6B, B1661–B1674 (Dec. 20, 1965).

Random-walk calculations of diffusion coefficients, R. E. Howard, Proc. 5th Intern. Symp. Reactivity of Solids, Munich, Germany, 1964, pp. 250–255 (Elsevier Publ. Co., Amsterdam, The Netherlands, 1965).

Relative transition probabilities for prominent Ni I and Ni II lines in the near-ultraviolet, G. D. Bell, D. R. Paquette, and W. L. Wiese, Astrophys. J. **143,** No. 2, 559–572 (Feb. 1966).

Resonances in the elastic and inelastic electron scattering from N_2, H. G. M. Heideman, C. E. Kuyatt, and G. E. Chamberlain, J. Chem. Phys. **44,** No. 1, 355–358 (Jan. 1, 1966).

Stress-strain properties of textile yarns subjected to rifle bullet impact, C. A. Fenstermaker and J. C. Smith, J. Appl. Polymer Sci. **1,** 125–146 (1965).

The correction factor for fast neutron reactions on sulfur and oxygen in the manganous-sulfate-bath calibration of neutron sources, W. M. Murphey, Nucl. Instr. Methods **37,** No. 1, 13–21 (1965).

The effect of humidity on ionization measurements with cavity and free-air chambers, T. P. Loftus, B. Petree, and J. T. Weaver, Jr., Radiology **86,** No. 1, 149 (Jan. 1966).

An investigation of concentric ring antennas with low sidelobes, C. O. Stearns and A. C. Stewart, IEEE Trans. Ant. Prop. **AP-13,** No. 6, 856–863 (Nov. 1965).

On radiation of electromagnetic and electroacoustic waves in plasma. Part II, J. R. Wait, Appl. Sci. Res. **12,** Sect. B, 130–138 (1965).

Adsorption isotherms for polymer chains adsorbed from theta solvents, C. A. J. Hoeve, J. Chem. Phys. **44,** No. 4, 1505–1509 (Feb. 15, 1966).

Critical scattering of the perfluoroheptane-iso-octane system, G. W. Brady, D. McIntyre, M. E. Myers, Jr., and A. M. Wims, J. Chem. Phys. **44,** No. 5, 2197–2198 (Mar. 1, 1966).

Polymer property-structure studies at the National Bureau of Standards; thermal stability, G. M. Kline, Proc. European Plastics Congress, Milano, Italy, Sept. 20–23, 1964.

Polymers and telomers of perfluoro-1, 4-pentadiene, J. E. Fearn, D. W. Brown, and L. A. Wall, J. Polymer Sci. **4,** Pt. A-1, 131–140 (1966).

The vaterite-type ABO_3 rare-earth borates, W. F. Bradley, D. L. Graf, and R. S. Roth, Acta Cryst. **20,** Pt. 2, 283–287 (Feb. 1966).

Accuracy and precision of standards and measurements, W. A. Wildhack, Acta Imeko **4,** 63–81 (1964).

Time-resolved electrical measurements in high current discharges, E. C. Cassidy, S. W. Zimmerman, and K. K. Neuman, Rev. Sci. Instr. **37,** No. 2, 210–214 (Feb. 1966).

Experimental verification of superachromatism, R. E. Stephens, J. Opt. Soc. Am. **56,** No. 2, 213–214 (Feb. 1966).

Force fields for the boron trihalides, I. W. Levin and S. Abramowitz, J. Chem. Phys. **43,** No. 12, 4213–4222 (Dec. 15, 1965).

Interpretation of Brillouin spectra, R. D. Mountain, J. Chem. Phys. **44,** No. 2, 832–833 (Jan. 15, 1966).

Isotopic effects in the ν_3 fundamental of matrix-isolated BCl_3, J. J. Comeford, S. Abramowitz, and I. W. Levin, J. Chem. Phys. **43,** No. 12, 4536–4537 (Dec. 15, 1965).

On the quantum-mechanical pair-correlation function of ^4He gas at low temperatures, S. Y. Larsen, K. Witte, and J. E. Kilpatrick, J. Chem. Phys. **44,** No. 1, 213–220 (Jan. 1, 1966).

Spectral distribution of scattered light in a simple fluid, R. D. Mountain, Rev. Mod. Phys. **38,** No. 1, 205–214 (Jan. 1966).

Spin-lattice relaxation in some rare-earth trichlorides, B. W. Mangum and R. P. Hudson, J. Chem. Phys. **44,** No. 2, 704–713 (Jan. 15, 1966).

The crystal structure of the strontium and lead tetraborates, $SrO \cdot 2B_2O_3$ and $PbO \cdot 2B_2O_3$, A. Perloff and S. Block, Acta Cryst. **20,** pt. 2, 274–279 (Feb. 1966).

Search methods used with transistor patent applications, J. R. Cornog and H. L. Bryan, Jr., IEEE Spectrum, pp. 116–121 (Feb. 1966).

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